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**2018** HIGHER SCHOOL CERTIFICATE COURSE MATERIALS

# Preliminary Mathematics Extension 1

## Parametric Equations

## Term 1 – Week 1

Name ………………………………………………………….

Class day and time ……………………………………………

Teacher name ……………………………………………...

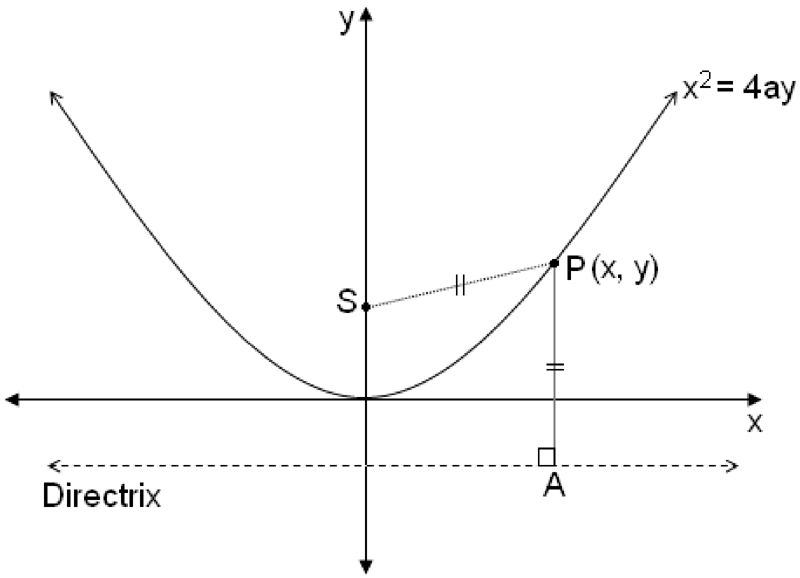
# Term 1 – Week 1 – Theory

## CARTESIAN REPRESENTATION OF THE PARABOLA $x^2 = \pm 4ay$ (REVISION):

A parabola is defined as the locus of all points that are equidistant from a fixed point and a given line. The point is known as the 'focus' and the line is known as the 'directrix'.

The parabola with focus $(0, a)$ and directrix $y = -a$ has Cartesian equation $x^2 = 4ay$.

The parabola with focus $(0, -a)$ and directrix $y = a$ has Cartesian equation $x^2 = -4ay$.




## EXAMPLE:

Find the locus of a point P that moves so that its distance from the point $(0, 2)$ is the same as its distance from the line $y = -2$.

SOLUTION:

Let the point be $P(x, y)$
Distance from $P$ to $(0, 2)$ is given by $\sqrt{(x-0)^2+(y-2)^2} = \sqrt{x^2+(y-2)^2}$
Distance from $P$ to $y = -2$ is given by $\left|\frac{0x+1y+2}{\sqrt{0^2+1^2}}\right| = \left|\frac{y+2}{1}\right| = |y+2|$

$\therefore \sqrt{x^2+(y-2)^2} = |y+2|$

$x^2+(y-2)^2 = (y+2)^2$

$x^2+y^2-4y+4 = y^2+4y+4$

$x^2 = 8y$

The locus is $x^2 = 8y$.

## PARAMETRIC EQUATION OF THE PARABOLA $x^2 = \pm 4ay$

The parabola $x^2 = 4ay$ can be represented by the parametric equations:

$x = 2at$ and $y = at^2$

where $t$ is known as a 'parameter'.

Therefore $P(2ap, ap^2)$ would represent a general point on the parabola, and $Q(2aq, aq^2)$ would represent another point on the parabola.

This means that substituting in any given value for the parameter will give us exactly one point on the parabola, and the locus of all such points is the parabola.

For example, consider the parabola $y = x^2$, or $x^2 = 4\left(\frac{1}{4}\right)y$ (i.e. $a = \frac{1}{4}$)

If we let $t = 2$, we obtain the point $\left(2\left(\frac{1}{4}\right)(2), \left(\frac{1}{4}\right)(2)^2\right) = (1, 1)$

If we let $t = -4$, we obtain the point $\left(2\left(\frac{1}{4}\right)(-4), \left(\frac{1}{4}\right)(-4)^2\right) = (-2, 4)$

The use of parametric representations allows important properties of the parabola and the equations of related curves (e.g. tangents, normals) to be expressed as functions of one parameter, $t$. This is helpful because it simplifies the algebra involved.

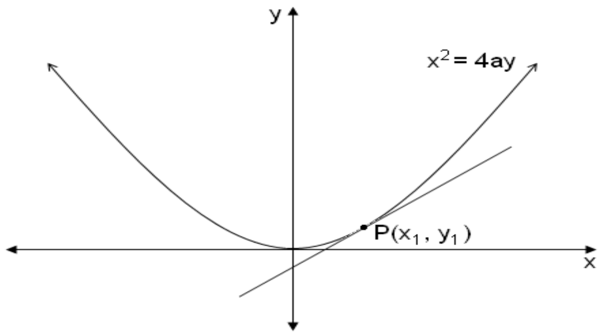
In the following sections, both the parametric and Cartesian representations are used to derive the equations of tangents, normals and chords.

Most questions will ask you to derive one or more of these equations in the first step, so that the results can be used to prove further properties. Therefore it is useful to know both the derivations and the results. These can be learnt simply through practicing on sample questions. There is no need to rote learn them.

### TANGENTS TO THE PARABOLA $x^2 = 4ay$

**1. Cartesian Representation**

Let $P(x_1, y_1)$ be a point on the parabola $x^2 = 4ay$.




$x^2 = 4ay$

Differentiating both sides with respect to $x$,

$$2x = 4a.\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{x}{2a}$$

$\therefore$ at the point $P(x_1, y_1)$,

$$\frac{dy}{dx} = \frac{x_1}{2a}$$

So the tangent is given by

$$y - y_1 = \frac{x_1}{2a}(x - x_1)$$

$$\therefore 2ay - 2ay_1 = xx_1 - x_1^2$$

$$\therefore xx_1 = 2ay - 2ay_1 + x_1^2$$

$$= 2ay - 2ay_1 + 4ay_1 \text{ as } (x_1, y_1) \text{ lies on } x^2 = 4ay.$$

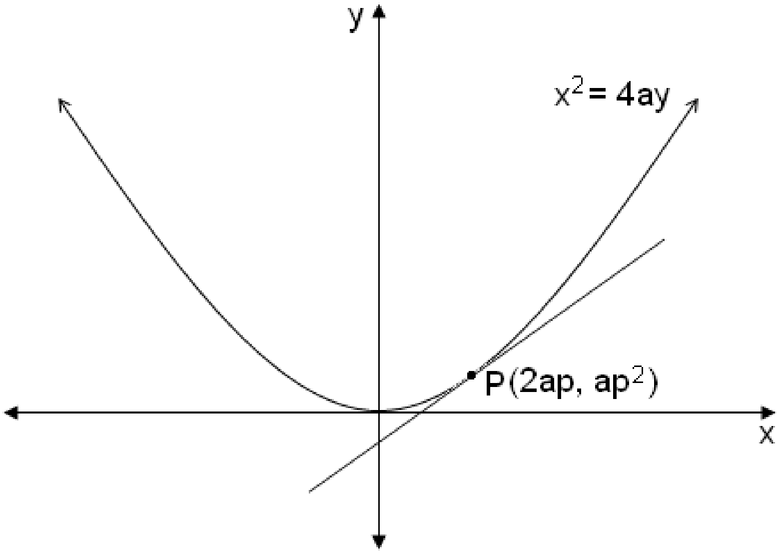
$$\therefore xx_1 = 2a(y + y_1)$$

$\therefore$ The tangent to the parabola $x^2 = 4ay$ at a point $P(x_1, y_1)$ is given by

$$\boxed{xx_1 = 2a(y + y_1)}$$

**2. Parametric Representation**

Let $P(2ap, ap^2)$ be a point on the parabola $x^2 = 4ay$ with parameter $p$.




$$x^2 = 4ay$$

Differentiating both sides with respect to $x$,

$$2x = 4a.\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{x}{2a}$$

$\therefore$ at the point $P(2ap, ap^2)$,

$$\frac{dy}{dx} = \frac{2ap}{2a} = p$$

So the tangent is given by

$$y - ap^2 = p(x - 2ap)$$

$$\therefore y - ap^2 = px - 2ap^2$$

$$\therefore y = px - ap^2$$

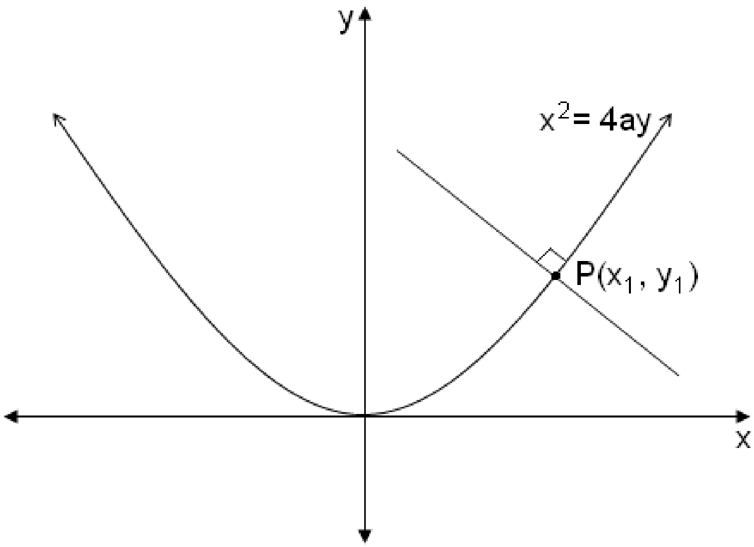
$\therefore$ The tangent to the parabola $x^2 = 4ay$ at a point $P(2ap, ap^2)$ is given by

$$\boxed{y = px - ap^2}$$

NORMALS TO THE PARABOLA $\boldsymbol{x^2 = 4ay}$

**1. Cartesian Representation**

Let $P(x_1,\ y_1)$ be a point on the parabola $x^2 = 4ay$.




$x^2 = 4ay$

Differentiating both sides with respect to $x$,

$$2x = 4a.\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{x}{2a}$$

$\therefore$ at the point $P(x_1, y_1)$, the gradient of the tangent to the curve is given by

$$\frac{dy}{dx} = \frac{x_1}{2a}$$

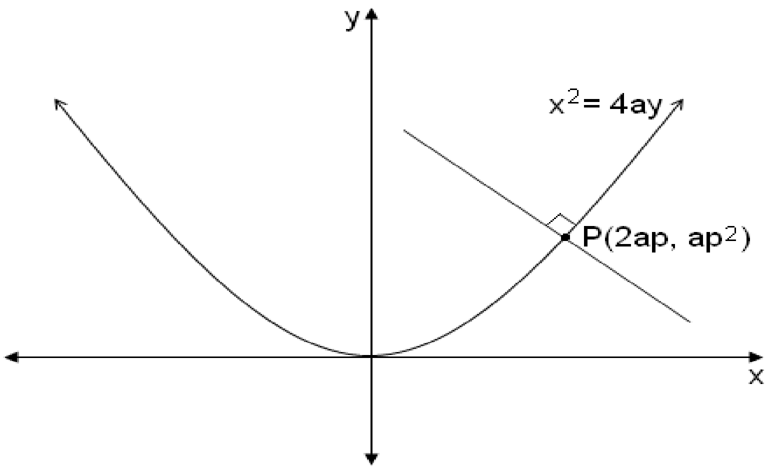
$\therefore$ the gradient of the normal is $-\frac{2a}{x_1}$

$\therefore$ the normal to the parabola $x^2 = 4ay$ at a point $P(x_1, y_1)$ is given by

$$\boxed{y - y_1 = -\frac{2a}{x_1}(x - x_1)}$$

**2. Parametric Representation**

Let $P(2ap, ap^2)$ be a point with parameter $p$ on the parabola $x^2 = 4ay$.




$$x^2 = 4ay$$

Differentiating both sides with respect to $x$,

$$2x = 4a.\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{x}{2a}$$

$\therefore$ at the point $P(2ap, ap^2)$, the gradient of the tangent to the curve is given by

$$\frac{dy}{dx} = \frac{2ap}{2a} = p$$

$\therefore$ the gradient of the normal is $-\frac{1}{p}$

So the normal is given by

$$y - ap^2 = -\frac{1}{p}(x - 2ap)$$

$$\therefore py - ap^3 = -x + 2ap$$

$$\therefore x + py = ap^3 + 2ap$$

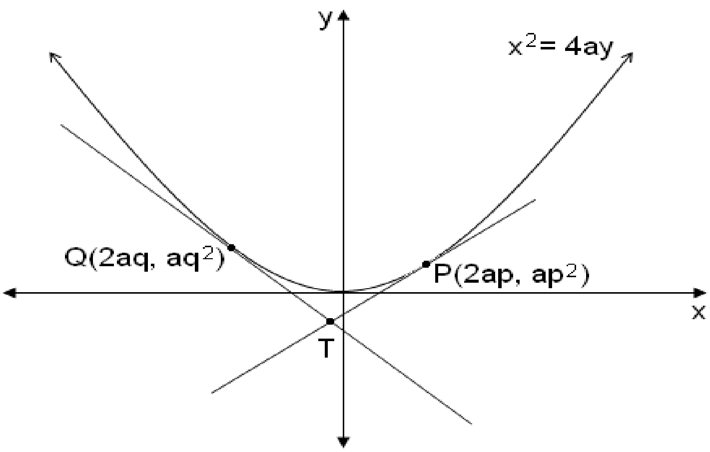
$\therefore$ the normal to the parabola $x^2 = 4ay$ at a point $P(2ap, ap^2)$ is given by

$$\boxed{x + py = ap^3 + 2ap}$$

INTERSECTION OF TANGENTS AND NORMALS OF THE PARABOLA $x^2 = 4ay$

Many problems require you to find the intersection of the tangents or normals at point $P(2ap, ap^2)$ and $Q(2aq, aq^2)$, and then to prove some property involving this intersection. Thus is it worthwhile to know both what the point is, and how to derive it.

**1. Intersection of Tangents**




Let the intersection be $T$

Equation of the tangent at $P$: $y = px - ap^2 ----(1)$

Equation of the tangent at $Q$: $y = qx - aq^2 ----(2)$

Solving $(1)$ and $(2)$ simultaneously,

$px - ap^2 = qx - aq^2$

$px - qx = ap^2 - aq^2$

$(p-q)x = a(p+q)(p-q)$

$\therefore x = a(p+q)$ as $p \neq q$ and so $p - q \neq 0$

$\therefore y = px - ap^2$

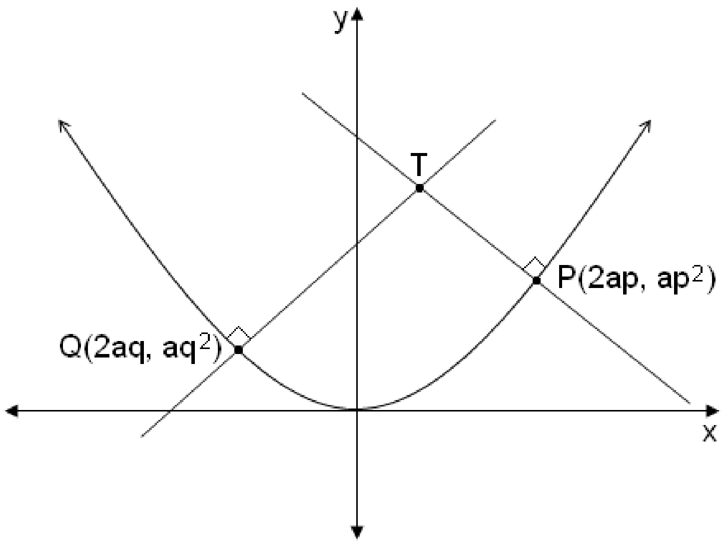
$= ap(p+q) - ap^2$

$= ap(p+q-p)$

$= apq$

$\therefore T[a(p+q), apq]$ is the intersection of the tangents at points $P$ and $Q$.

**2. Intersection of Normals**




Let the intersection be $T$

Equation of normal at $P$: $x + py = ap^3 + 2ap ----(1)$

Equation of normal at $Q$: $x + qy = aq^3 + 2aq ----(2)$

Subtracting (2) from (1):

$py - qy = ap^3 + 2ap - aq^3 - 2aq$

$(p - q)y = a\big(p^3 - q^3 + 2(p - q)\big)$

$= a(p - q)(p^2 + pq + q^2 + 2)$

$\therefore y = a(p^2 + pq + q^2 + 2)$ as $p \neq q$ and so $p - q \neq 0$

$\therefore x = ap^3 + 2ap - py$

$= ap^3 + 2ap - ap(p^2 + pq + q^2 + 2)$

$= ap^3 + 2ap - ap^3 - ap^2q - apq^2 - 2ap$

$= -ap^2q - apq^2$

$= -apq(p + q)$

$\therefore T[-apq(p + q), a(p^2 + pq + q^2 + 2)]$ is the intersection of the normals at points $P$ and $Q$.

# Term 1 – Week 1 – Homework

1. By using differentiation, find the equation of the tangent to the parabola at the indicated points:
   a) $x = 2t$, $y = t^2$ at the point where $t = 1$
   b) $x = 4t$, $y = 2t^2$ at the point where $t = -\frac{1}{2}$
   c) $x = t$, $y = \frac{1}{2}t^2$ at the point where $t = 4$
   d) $x = 2at$, $y = at^2$ at the point where $t = 3$
   e) $x^2 = 4y$ at the point $(-2,1)$
   f) $x^2 = -8y$ at the point $\left(-2,-\frac{1}{2}\right)$
   g) $x^2 = 6y$ at the point $(6,6)$
   h) $x^2 = 4ay$ at the point $(x_1, y_1)$

2.

(i) Find the equation of the tangent to the parabola $x^2 = 8y$ at the point $(4t, 2t^2)$

(ii) Hence determine all tangents to the parabola that pass through the point $(1, -1)$

3. $P(2p, p^2)$ and $Q\left(2\left(\frac{1}{p}\right), \left(\frac{1}{p}\right)^2\right)$ are two variable points on the parabola $x^2 = 4y$. The tangents at P and Q intersect at a point T.

(i) Find the equation of the tangent to the parabola at P.

(ii) Determine the coordinates of T.

(iii) Hence find the Cartesian equation of the locus of T.

4. The line $ax+by=1$ is tangent to the parabola $x^2=-4y$. Find the conditions on $a$ and $b$.

5. $P(2ap, ap^2)$ is a variable point on the parabola $x^2=4ay$. The tangent at P intersects the x-axis at A and the y-axis at B. C is the fourth vertex of rectangle OACB.
   (i) Find the coordinates of C in terms of $p$.
   (ii) Hence show that the locus of C is a parabola and state its vertex and focus.

6. $P(2ap, ap^2)$ is a variable point on the parabola $x^2 = 4ay$. T is the foot of the perpendicular drawn from the focus to the tangent at P. Find the Cartesian equation of the locus of T.

7. By using differentiation, find the equation of the normal to the parabola at the indicated points:
   a) $x = 2t$, $y = t^2$ at the point where $t = -2$
   b) $x = 6t$, $y = 3t^2$ at the point where $t = 4$
   c) $x = t$, $y = \frac{1}{2}t^2$ at the point where $t = 1$
   d) $x = 2at$, $y = at^2$ at the point where $t = p$
   e) $x^2 = 4y$ at the point $(2,1)$
   f) $x^2 = -y$ at the point $(1,-1)$
   g) $x^2 = 12y$ at the point $(-6,3)$
   h) $x^2 = \frac{1}{4}y$ at the point $(x_1, y_1)$

8.

(i) Find the equation of the normal to the parabola $x = 2at$, $y = at^2$ at the point where $t = p$.

(ii) The normal intersects the $x$-axis at A and the $y$-axis at B. Find the coordinates of A and B.

(iii) Hence determine the area of $\Delta AOB$

9.

(i) Find the equation of the parabola that is symmetrical about the y-axis and passes through the points $(1,1)$ and $(-4,4)$.

(ii) Find the normal to the parabola at the point $(1,1)$.

10. $P\left(2ap, ap^2\right)$ is a variable point on the parabola $x^2 = 4ay$. The normal at P intersects the y-axis at T. M is the midpoint of PT.
   (i) Find the coordinates of T.
   (ii) Hence find the coordinates of M and determine the Cartesian equation of the locus of M.

**End of Homework**