Year 11 Mathematics Advanced
Locus - Parabola
Term 1 – Week 2

Name .........................................................................................

Class day and time .................................................................

Teacher name ........................................................................
A PARABOLA:

A parabola may be defined as the locus of a point \( P(x, y) \) whose distance from a given fixed point equals its distance from a given fixed line. The fixed point is known as the focus and the fixed line is known as the directrix.

The vertex is the minimum or maximum point of the parabola. The axis of symmetry is a line which bisects the parabola. The focal length is the distance between the vertex and the focus or the vertex and the directrix.
THE GENERAL EQUATION OF THE LOCUS OF A PARABOLA WITH VERTEX AT THE ORIGIN:

CASE ONE (EXAMPLE):
Find the locus of a point \( P(x, y) \) such that its distance from the point \( F(0, a) \) is equal to its distance from the line \( y = -a \).

SOLUTION:

![Diagram of parabola with focus and directrix](image)

Given the distance of \( FP = AP \), so use the distance formula:

\[
FP = AP \\
\sqrt{(x - 0)^2 + (y - a)^2} = \sqrt{(x - x)^2 + (y - (-a))^2} \\
x^2 + (y - a)^2 = 0^2 + (y + a)^2 \\
x^2 + y^2 - 2ay + a^2 = y^2 + 2ay + a^2
\]

\[
\therefore x^2 = 4ay
\]
CASE TWO (EXAMPLE):

Find the locus of a point $P(x, y)$ such that its distance from the point $F(0, -a)$ is equal to its distance from the line $y = a$.

SOLUTION:

Given the distance of $FP = AP$, so use the distance formula:

$$FP = AP$$

$$\sqrt{(x - 0)^2 + (y - a)^2} = \sqrt{(x - x)^2 + (y - a)^2}$$

$$x^2 + (y + a)^2 = 0^2 + (y - a)^2$$

$$x^2 + y^2 + 2ay + a^2 = y^2 - 2ay + a^2$$

$$\therefore x^2 = -4ay$$
CASE THREE (EXAMPLE):

Find the locus of a point $P(x, y)$ such that its distance from the point $F(a, 0)$ is equal to its distance from the line $x = -a$.

**SOLUTION:**

Given the distance of $FP = AP$, so use the distance formula:

$$FP = AP$$

$$\sqrt{(x - a)^2 + (y - 0)^2} = \sqrt{(x + a)^2 + (y - 0)^2}$$

$$(x - a)^2 + y^2 = (x + a)^2 + 0^2$$

$$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$$

$$\therefore y^2 = 4ax$$
CASE FOUR (EXAMPLE):

Find the locus of a point \( P(x, y) \) such that its distance from the point \( F(-a, 0) \) is equal to its distance from the line \( x = a \).

**SOLUTION:**

Given the distance of \( FP = AP \), so use the distance formula:

\[
FP = AP \\
\sqrt{(x - a)^2 + (y - 0)^2} = \sqrt{(x - a)^2 + (y - y)^2} \\
(x + a)^2 + y^2 = (x - a)^2 + 0^2 \\
x^2 + 2ax + a^2 + y^2 = x^2 - 2ax + a^2 \\
\therefore y^2 = -4ax
\]
EXAMPLE:

Sketch the parabola with equation $x^2 = -16y$ and then find:

(i) The focal length,
(ii) The coordinates of the vertex,
(iii) The coordinates of the focus,
(iv) The equation of the directrix,
(v) The equation of the axis of symmetry.

SOLUTION:

\[ x^2 = -16y \]

(i) The focal length:
\[ 4a = 16 \]
\[ \therefore a = 4 \]
\[ \therefore \text{The focal length is 4 units.} \]

(ii) The coordinates of the vertex is \((0, 0)\).

(iii) The coordinates of the focus is \((0, -4)\).

(iv) The equation of the directrix is \(y = 4\).

(v) The equation of the axis of symmetry is \(x = 0\).
THE GENERAL EQUATION OF THE LOCUS OF A PARABOLA WITH VERTEX NOT AT THE ORIGIN:

CASE ONE (EXAMPLE):

Find the locus of a point \( P(x, y) \) such that its distance from the point \( F(h, k + a) \) is equal to its distance from the line \( y = k - a \).

SOLUTION:

Given the distance of \( FP = AP \), so use the distance formula:

\[
FP = AP \\
\sqrt{(x-h)^2 + (y-(k+a))^2} = \sqrt{(x-x)^2 + (y-(k-a))^2} \\
(x-h)^2 + (y-(k+a))^2 = (y-(k-a))^2 \\
(x-h)^2 + y^2 - 2(k+a)y + (k+a)^2 = y^2 - 2(k-a)y + (k-a)^2 \\
(x-h)^2 + y^2 - 2ky - 2ay + k^2 + 2ak + a^2 = y^2 - 2ky + 2ay + k^2 - 2ak + a^2 \\
(x-h)^2 = 4ay - 4ak \\
\therefore (x-h)^2 = 4a(y-k)
CASE TWO (EXAMPLE):

Find the locus of a point \( P(x, y) \) such that its distance from the point \( F(h, k-a) \) is equal to its distance from the line \( y = k + a \).

**SOLUTION:**

Given the distance of \( FP = AP \), so use the distance formula:

\[
FP = AP
\]

\[
\sqrt{(x-h)^2 + (y - (k-a))^2} = \sqrt{(x-x)^2 + (y - (k+a))^2}
\]

\[
(x-h)^2 + (y - (k-a))^2 = (y - (k+a))^2
\]

\[
(x-h)^2 + y^2 -2(k-a)y + (k-a)^2 = y^2 -2(k+a)y + (k+a)^2
\]

\[
(x-h)^2 + y^2 -2ky + 2ay + k^2 -2ak + a^2 = y^2 -2ky -2ay + k^2 + 2ak + a^2
\]

\[
(x-h)^2 = -4ay + 4ak
\]

\[
\therefore (x-h)^2 = -4a(y - k)
\]
CASE THREE (EXAMPLE):

Find the locus of a point \( P(x, y) \) such that its distance from the point \( F(h + a, k) \) is equal to its distance from the line \( x = h - a \).

**SOLUTION:**

Given the distance of \( FP = AP \), so use the distance formula:

\[
FP = AP
\]

\[
\sqrt{(x - (h + a))^2 + (y - k)^2} = \sqrt{(x - (h - a))^2 + (y - y)^2}
\]

\[
(x - (h + a))^2 + (y - k)^2 = (x - (h - a))^2
\]

\[
x^2 - 2(h + a)x + (h + a)^2 + (y - k)^2 = x^2 - 2(h - a)x + (h - a)^2
\]

\[
x^2 - 2hx - 2ax + h^2 + 2ah + a^2 + (y - k)^2 = x^2 - 2hx + 2ax + h^2 - 2ah + a^2
\]

\[
(y - k)^2 = 4ax - 4ah
\]

\[
\therefore (y - k)^2 = 4a(x - h)
\]
CASE FOUR (EXAMPLE):

Find the locus of a point \( P(x, y) \) such that its distance from the point \( F(h - a, k) \) is equal to its distance from the line \( x = h + a \).

SOLUTION:

Given the distance of \( FP = AP \), so use the distance formula:

\[
FP = AP
\]

\[
\sqrt{(x - (h - a))^2 + (y - k)^2} = \sqrt{(x - (h + a))^2 + (y - y)^2}
\]

\[
(x - (h - a))^2 + (y - k)^2 = (x - (h + a))^2
\]

\[
x^2 - 2(h - a)x + (h - a)^2 + (y - k)^2 = x^2 - 2(h + a)x + (h + a)^2
\]

\[
x^2 - 2hx + 2ax + h^2 - 2ah + a^2 + (y - k)^2 = x^2 - 2hx - 2ax + h^2 + 2ah + a^2
\]

\[
(y - k)^2 = -4ax + 4ah
\]

\[
\therefore (y - k)^2 = -4a(x - h)
\]
EXAMPLE:

Find the equation of the locus of a point \( P(x, y) \) such that its distance from the point \( F(3, 2) \) is equal to its distance from the line \( y = -6 \).

SOLUTION:

Firstly, sketch a diagram.

Given the distance of \( FP = AP \), so use the distance formula:

\[
FP = AP
\]

\[
\sqrt{(x - 3)^2 + (y - 2)^2} = \sqrt{(x - x)^2 + (y + 6)^2}
\]

\[
(x - 3)^2 + (y - 2)^2 = (y + 6)^2
\]

\[
(x - 3)^2 + y^2 - 4y + 4 = y^2 + 12y + 36
\]

\[
(x - 3)^2 = 16y + 32
\]

\[
\therefore (x - 3)^2 = 16(y + 2)
\]
EXAMPLE:

For the parabola with equation $2y^2 - 8y + x + 9 = 0$, find:

(i) The coordinates of the vertex and focus.
(ii) The equation of the directrix and the axis of symmetry.
(iii) Hence, sketch the parabola.

SOLUTION:

(i)

\[
2y^2 - 8y + x + 9 = 0
\]
\[
y^2 - 4y + \frac{1}{2}x + \frac{9}{2} = 0
\]
\[
y^2 - 4y + (-2)^2 = -\frac{1}{2}x - \frac{9}{2} + (-2)^2
\]
\[
(y - 2)^2 = -\frac{1}{2}x - \frac{1}{2}
\]
\[
(y - 2)^2 = -\frac{1}{2}(x + 1)
\]

The focal length:

\[
4a = \frac{1}{2}
\]
\[
a = \frac{1}{8}
\]

\[-\] The vertex is $(-1, 2)$ and the focus is $\left(-1 \frac{1}{8}, 2\right)$.

(ii) The equation of the directrix is $x = -\frac{7}{8}$ and the axis of symmetry is $y = 2$.

(iii)
EXAMPLE:

Find the equation of the parabola with vertex \((2, 3)\) and the equation of the directrix is \(x = 3\).

SOLUTION:

Firstly, sketch a diagram.

Hence, the focal length is 1.

\[
(y - k)^2 = -4a(x - h)
\]

\[
(y - 3)^2 = -4(1)(x - 2)
\]

\[
\therefore (y - 3)^2 = -4(x - 2)
\]
Term 1 – Week 2 – Homework

LOCUS OF A PARABOLA:

1. Find the co-ordinates of the vertex of the parabola $y = x^2 + 4x - 3$. (2 marks)

2. For the parabola $y = (x - 4)^2 - 5$ write down:
   (i) The co-ordinates of the vertex. (1 mark)
   (ii) The minimum value of the function. (1 mark)
   (iii) The co-ordinates of the focus. (1 mark)
3. Write down the equation of the directrix of the parabola: \( x^2 = -8y \). (1 mark)

4. Find the vertex and focus of \((x + 2)^2 = -12y + 6\). (2 marks)
Given the parabola $12y = x^2 - 4x + 52$ express it in the form $(x - x_1)^2 = 4a(y - y_1)$. Find:

(i) The co-ordinates of the vertex. (1 mark)
(ii) The focal length. (1 mark)
(iii) The co-ordinates of the focus. (1 mark)
(iv) The equation of the directrix. (1 mark)
(v) The equation of the axis of the parabola. (1 mark)
6. For the parabola \( y = \frac{1}{9}x^2 - x + 3 \)
   
   (i) Find the co-ordinates of the vertex and focus. (2 marks)
   
   (ii) Find the equation of the normal to the parabola at the point where \( x = -4 \). Write your answer in general form. (1 mark)
   
   (iii) Find the point on the parabola at which the tangent is parallel to \( y = 3x + 1 \). (1 mark)
7. Consider the parabola $x^2 = 4(y - 5)$
   (i) Write down the co-ordinates of the vertex. (1 mark)
   (ii) What are the co-ordinates of the focus. (1 mark)
   (iii) Sketch the parabola $x^2 = 4(y - 5)$. (1 mark)
8. Given the parabola $(x + 2)^2 = 8(y - 1)$ write down:
   (i) The co-ordinates of the focus. (1 mark)
   (ii) The equation of the directrix. (1 mark)

9. A parabola has the equation $x^2 = 12(8 - y)$. What is the equation of the directrix? (1 mark)
10. The parabola $P$ has equation $x^2 = 16(y + 1)$

(i) Draw a neat $\frac{1}{3}$ page sketch of $P$ and clearly indicate on it:
   a) The equation of the directrix. (1 mark)
   b) The co-ordinates of its focus. (1 mark)
   c) The co-ordinates of its vertex. (1 mark)

(ii) Find the co-ordinates of the point where $P$ cuts the $x$ axis. (1 mark)
11. Sketch the locus of the point \( P(x, y) \) which moves so that it is equidistant from the point \((1,2)\) and the line \( y = -2 \). Describe the locus of \( P \) geometrically and find its equation. (3 marks)
12. The equation \((x - 2)^2 = 6 \left(y - \frac{1}{2}\right)\) represents a parabola.
   (i) Write down the co-ordinates of the vertex. (1 mark)
   (ii) Find the focal length. (1 mark)
   (iii) Find the \(x\) intercept of the parabola. (1 mark)
   (iv) Sketch the graph of the parabola, show the focus and directrix on your diagram. (3 marks)