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2018 HIGHER SCHOOL CERTIFICATE COURSE MATERIALS

Year 11 Mathematics Advanced

Locus - Parabola

Term 1 – Week 2

Name ……………………………………………………….

Class day and time ……………………………………………

Teacher name ………………………………………………...

Term 1 – Week 2 – Theory

A PARABOLA:

A parabola may be defined as the locus of a point $P(x, y)$ whose distance from a given fixed point equals its distance from a given fixed line. The fixed point is known as the **focus** and the fixed line is known as the **directrix**.

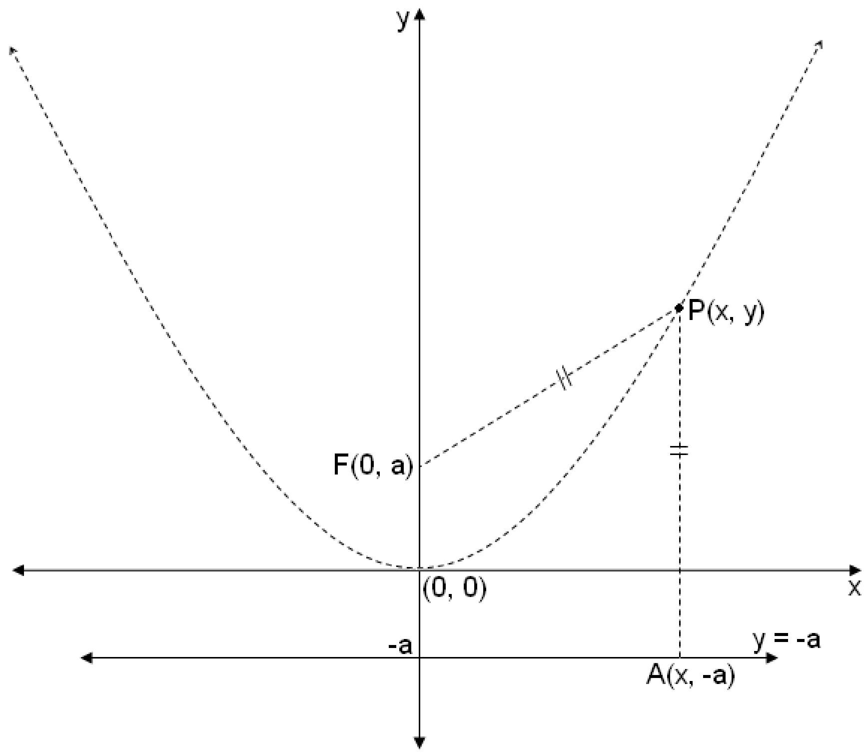
The **vertex** is the minimum or maximum point of the parabola. The **axis of symmetry** is a line which bisects the parabola. The **focal length** is the distance between the vertex and the focus or the vertex and the directrix.

THE GENERAL EQUATION OF THE LOCUS OF A PARABOLA WITH VERTEX AT THE ORIGIN:

CASE ONE (EXAMPLE):

Find the locus of a point $P(x, y)$ such that its distance from the point $F(0, a)$ is equal to its distance from the line $y = -a$.

SOLUTION:



Given the distance of $FP = AP$, so use the distance formula:

$$FP = AP$$

$$\sqrt{(x-0)^2 + (y-a)^2} = \sqrt{(x-x)^2 + (y--a)^2}$$

$$x^2 + (y-a)^2 = 0^2 + (y+a)^2$$

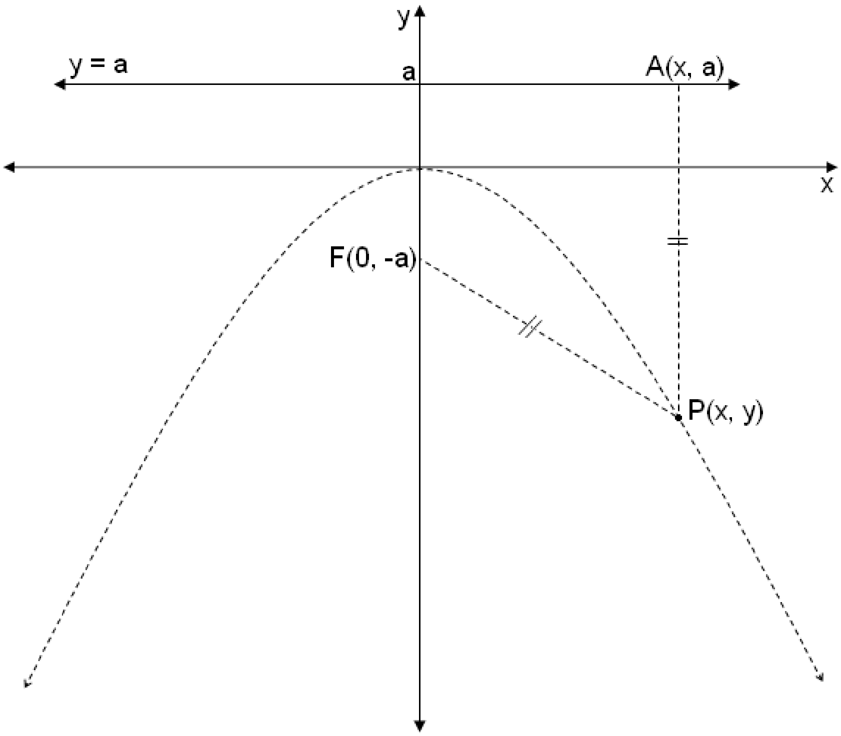
$$x^2 + y^2 - 2ay + a^2 = y^2 + 2ay + a^2$$

$$\boxed{\therefore x^2 = 4ay}$$

CASE TWO (EXAMPLE):

Find the locus of a point $P(x, y)$ such that its distance from the point $F(0, -a)$ is equal to its distance from the line $y = a$.

SOLUTION:



Given the distance of $FP = AP$, so use the distance formula:

$$FP = AP$$

$$\sqrt{(x-0)^2 + (y - -a)^2} = \sqrt{(x-x)^2 + (y-a)^2}$$

$$x^2 + (y+a)^2 = 0^2 + (y-a)^2$$

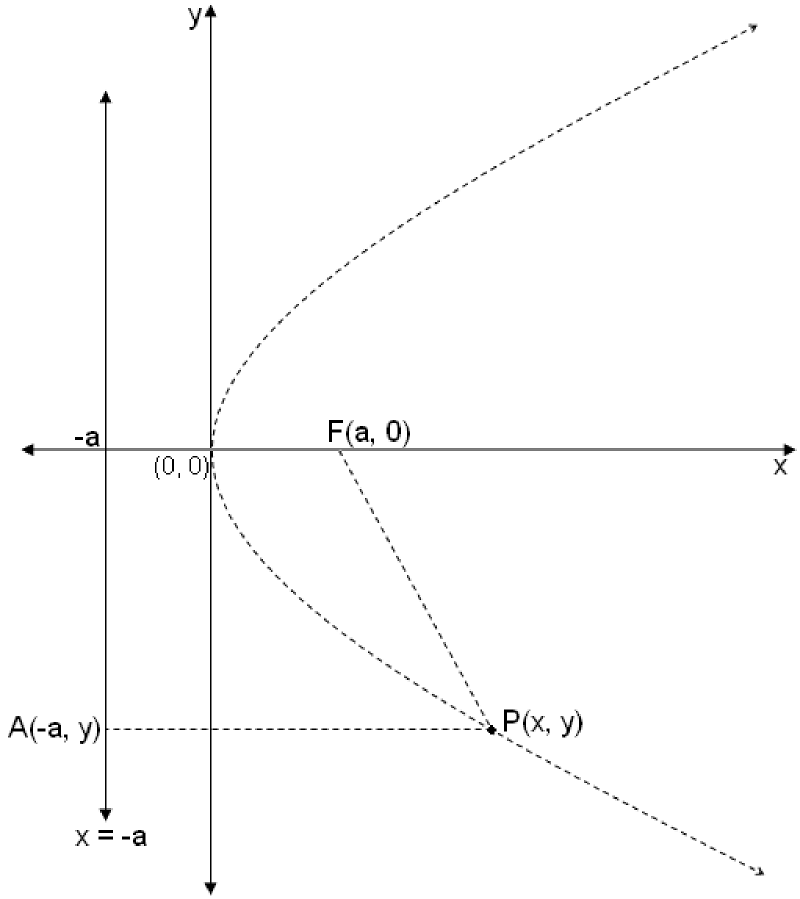
$$x^2 + y^2 + 2ay + a^2 = y^2 - 2ay + a^2$$

$$\therefore x^2 = -4ay$$

CASE THREE (EXAMPLE):

Find the locus of a point $P(x, y)$ such that its distance from the point $F(a, 0)$ is equal to its distance from the line $x = -a$.

SOLUTION:



Given the distance of $FP = AP$, so use the distance formula:

$$FP = AP$$

$$\sqrt{(x-a)^2 + (y-0)^2} = \sqrt{(x - -a)^2 + (y-y)^2}$$

$$(x-a)^2 + y^2 = (x+a)^2 + 0^2$$

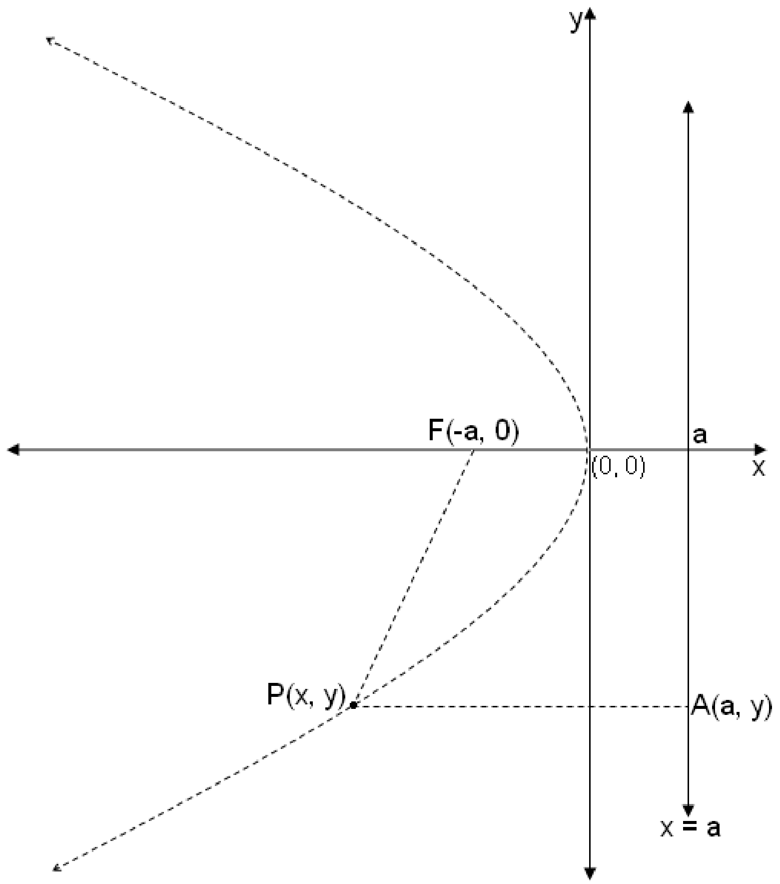
$$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$$

$$\boxed{\therefore y^2 = 4ax}$$

CASE FOUR (EXAMPLE):

Find the locus of a point $P(x, y)$ such that its distance from the point $F(-a, 0)$ is equal to its distance from the line $x = a$.

SOLUTION:



Given the distance of $FP = AP$, so use the distance formula:

$$FP = AP$$

$$\sqrt{(x - -a)^2 + (y - 0)^2} = \sqrt{(x - a)^2 + (y - y)^2}$$

$$(x + a)^2 + y^2 = (x - a)^2 + 0^2$$

$$x^2 + 2ax + a^2 + y^2 = x^2 - 2ax + a^2$$

$$\therefore y^2 = -4ax$$

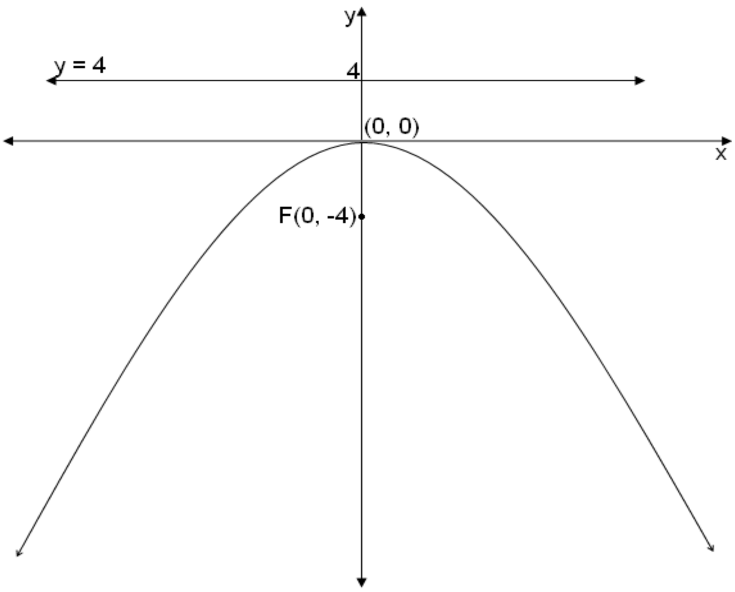
EXAMPLE:

Sketch the parabola with equation $x^2 = -16y$ and then find:

(i) The focal length,
(ii) The coordinates of the vertex,
(iii) The coordinates of the focus,
(iv) The equation of the directrix,
(v) The equation of the axis of symmetry.

SOLUTION:

$$x^2 = -16y$$



(i) The focal length:
$4a = 16$
$\therefore a = 4$
$\therefore$ The focal length is 4 units.

(ii) The coordinates of the vertex is $(0, 0)$.

(iii) The coordinates of the focus is $(0, -4)$.

(iv) The equation of the directrix is $y = 4$.

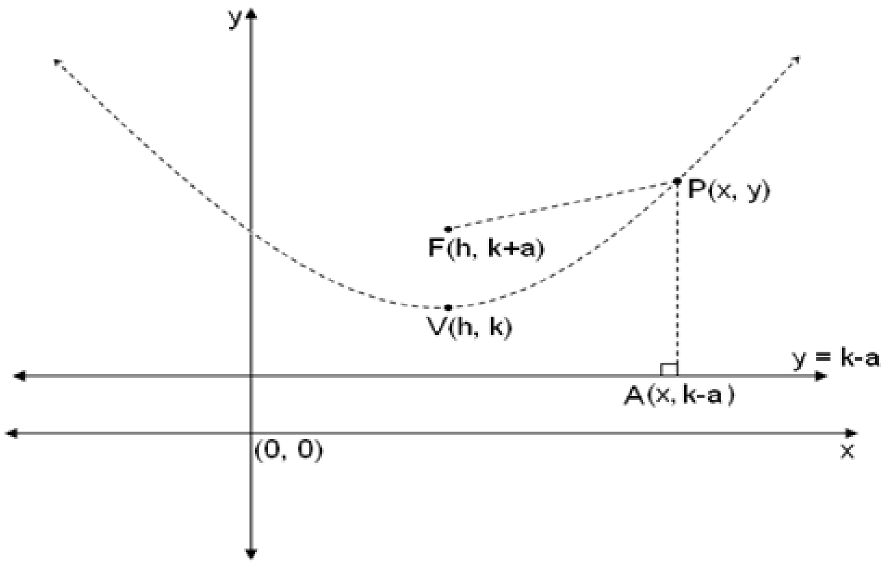
(v) The equation of the axis of symmetry is $x = 0$.

THE GENERAL EQUATION OF THE LOCUS OF A PARABOLA WITH VERTEX NOT AT THE ORIGIN:

CASE ONE (EXAMPLE):

Find the locus of a point $P(x, y)$ such that its distance from the point $F(h, k + a)$ is equal to its distance from the line $y = k - a$.

SOLUTION:



Given the distance of $FP = AP$, so use the distance formula:

$$FP = AP$$

$$\sqrt{(x-h)^2 + \big(y-(k+a)\big)^2} = \sqrt{(x-x)^2 + \big(y-(k-a)\big)^2}$$

$$(x-h)^2 + \big(y-(k+a)\big)^2 = \big(y-(k-a)\big)^2$$

$$(x-h)^2 + y^2 - 2(k+a)y + (k+a)^2 = y^2 - 2(k-a)y + (k-a)^2$$

$$(x-h)^2 + y^2 - 2ky - 2ay + k^2 + 2ak + a^2 = y^2 - 2ky + 2ay + k^2 - 2ak + a^2$$

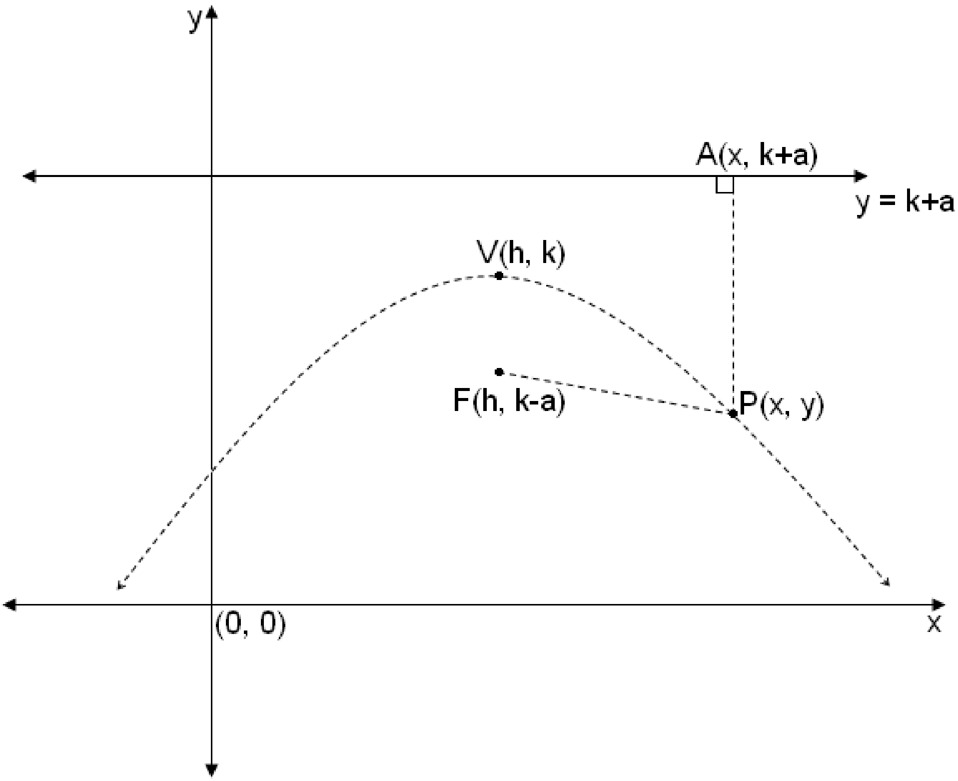
$$(x-h)^2 = 4ay - 4ak$$

$$\therefore (x-h)^2 = 4a(y-k)$$

CASE TWO (EXAMPLE):

Find the locus of a point $P(x, y)$ such that its distance from the point $F(h, k-a)$ is equal to its distance from the line $y = k + a$.

SOLUTION:



Given the distance of $FP = AP$, so use the distance formula:

$$FP = AP$$

$$\sqrt{(x-h)^2 + \big(y-(k-a)\big)^2} = \sqrt{(x-x)^2 + \big(y-(k+a)\big)^2}$$

$$(x-h)^2 + \big(y-(k-a)\big)^2 = \big(y-(k+a)\big)^2$$

$$(x-h)^2 + y^2 - 2(k-a)y + (k-a)^2 = y^2 - 2(k+a)y + (k+a)^2$$

$$(x-h)^2 + y^2 - 2ky + 2ay + k^2 - 2ak + a^2 = y^2 - 2ky - 2ay + k^2 + 2ak + a^2$$

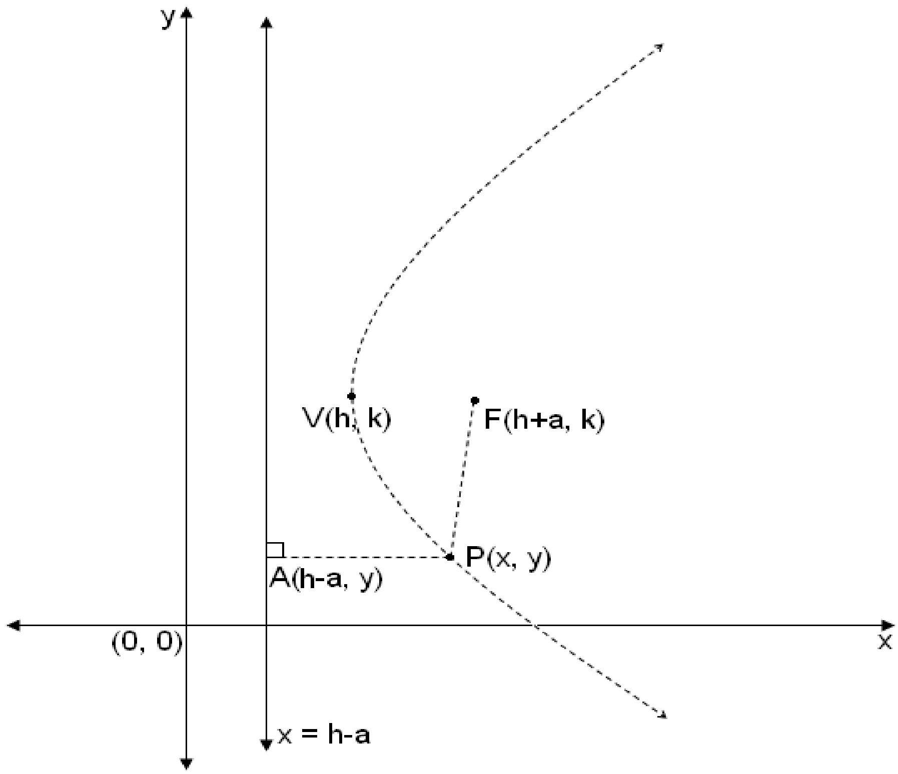
$$(x-h)^2 = -4ay + 4ak$$

$$\boxed{\therefore (x-h)^2 = -4a(y-k)}$$

CASE THREE (EXAMPLE):

Find the locus of a point $P(x, y)$ such that its distance from the point $F(h + a, k)$ is equal to its distance from the line $x = h - a$.

SOLUTION:



Given the distance of $FP = AP$, so use the distance formula:

$$FP = AP$$

$$\sqrt{\left(x-(h+a)\right)^2+(y-k)^2}=\sqrt{\left(x-(h-a)\right)^2+(y-y)^2}$$

$$\left(x-(h+a)\right)^2+(y-k)^2=\left(x-(h-a)\right)^2$$

$$x^2-2(h+a)x+(h+a)^2+(y-k)^2=x^2-2(h-a)x+(h-a)^2$$

$$x^2-2hx-2ax+h^2+2ah+a^2+(y-k)^2=x^2-2hx+2ax+h^2-2ah+a^2$$

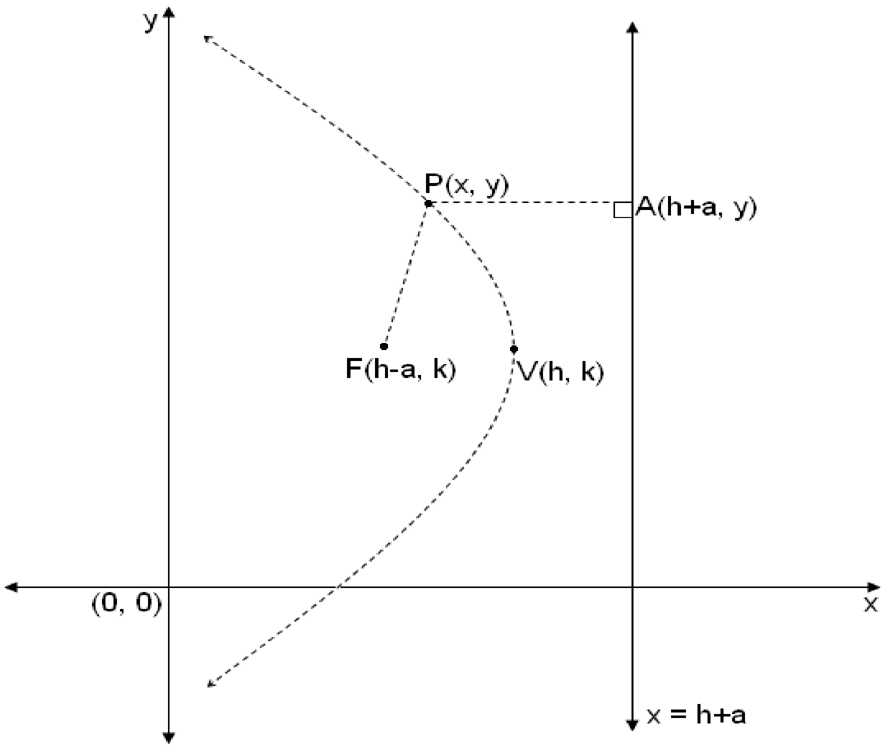
$$(y-k)^2=4ax-4ah$$

$$\boxed{\therefore (y-k)^2=4a(x-h)}$$

CASE FOUR (EXAMPLE):

Find the locus of a point $P(x, y)$ such that its distance from the point $F(h-a, k)$ is equal to its distance from the line $x = h + a$.

SOLUTION:



Given the distance of $FP = AP$, so use the distance formula:

$$FP = AP$$

$$\sqrt{\left(x-(h-a)\right)^2+(y-k)^2}=\sqrt{\left(x-(h+a)\right)^2+(y-y)^2}$$

$$\left(x-(h-a)\right)^2+(y-k)^2=\left(x-(h+a)\right)^2$$

$$x^2-2(h-a)x+(h-a)^2+(y-k)^2=x^2-2(h+a)x+(h+a)^2$$

$$x^2-2hx+2ax+h^2-2ah+a^2+(y-k)^2=x^2-2hx-2ax+h^2+2ah+a^2$$

$$(y-k)^2=-4ax+4ah$$

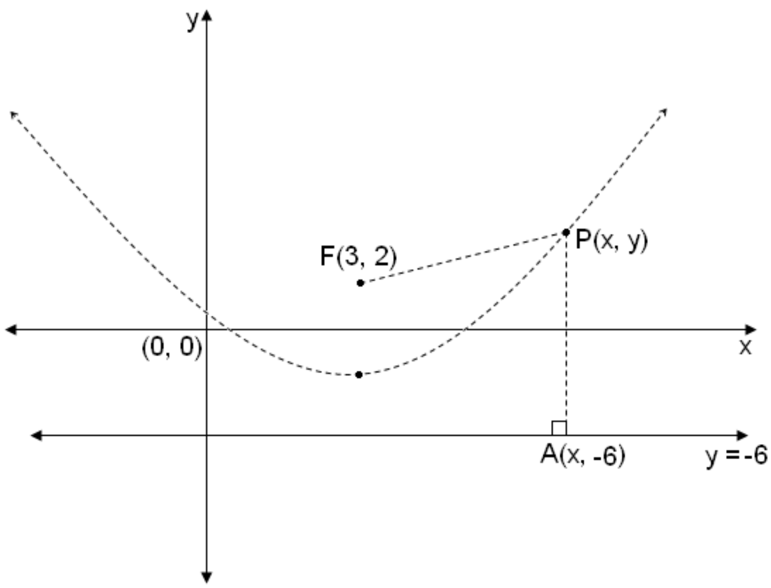
$$\boxed{\therefore (y-k)^2=-4a(x-h)}$$

EXAMPLE:

Find the equation of the locus of a point $P(x, y)$ such that its distance from the point $F(3, 2)$ is equal to its distance from the line $y = -6$.

SOLUTION:

Firstly, sketch a diagram.



Given the distance of $FP = AP$, so use the distance formula:

$$FP = AP$$

$$\sqrt{(x-3)^2 + (y-2)^2} = \sqrt{(x-x)^2 + (y+6)^2}$$

$$(x-3)^2 + (y-2)^2 = (y+6)^2$$

$$(x-3)^2 + y^2 - 4y + 4 = y^2 + 12y + 36$$

$$(x-3)^2 = 16y + 32$$

$$\therefore (x-3)^2 = 16(y+2)$$

EXAMPLE:

For the parabola with equation $2y^2 - 8y + x + 9 = 0$, find:

(i) The coordinates of the vertex and focus.
(ii) The equation of the directrix and the axis of symmetry.
(iii) Hence, sketch the parabola.

SOLUTION:

(i)

$$\begin{aligned} 2y^2 - 8y + x + 9 &= 0 \\ y^2 - 4y + \frac{1}{2}x + \frac{9}{2} &= 0 \\ y^2 - 4y + (-2)^2 &= -\frac{1}{2}x - \frac{9}{2} + (-2)^2 \\ (y-2)^2 &= -\frac{1}{2}x - \frac{1}{2} \\ (y-2)^2 &= -\frac{1}{2}(x+1) \end{aligned}$$

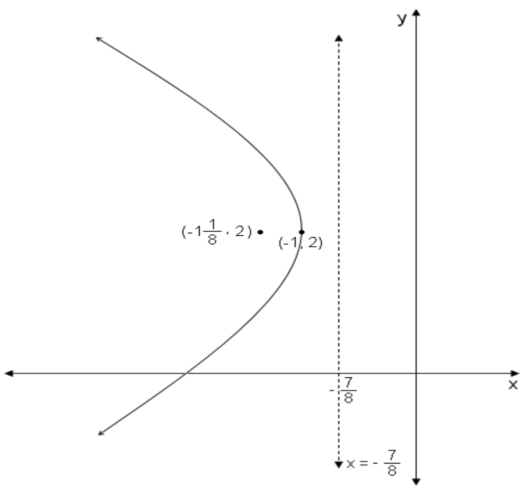
The focal length:

$$\begin{aligned} 4a &= \frac{1}{2} \\ a &= \frac{1}{8} \end{aligned}$$

$\therefore$ The vertex is $(-1, 2)$ and the focus is $\left(-1\frac{1}{8}, 2\right)$.

(ii) The equation of the directrix is $x = -\frac{7}{8}$ and the axis of symmetry is $y = 2$.

(iii)

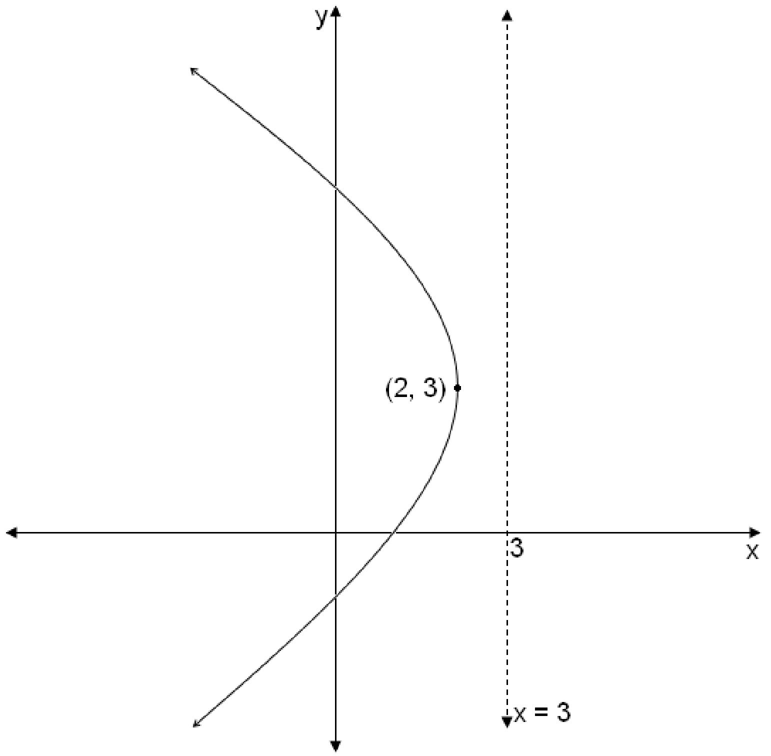


EXAMPLE:

Find the equation of the parabola with vertex $(2, 3)$ and the equation of the directrix is $x = 3$.

SOLUTION:

Firstly, sketch a diagram.



Hence, the focal length is 1.

$$(y - k)^2 = -4a(x - h)$$

$$(y - 3)^2 = -4(1)(x - 2)$$

$$\therefore (y - 3)^2 = -4(x - 2)$$

Term 1 – Week 2 – Homework

LOCUS OF A PARABOLA:

1. Find the co-ordinates of the vertex of the parabola $y = x^2 + 4x - 3$. (2 marks)

2. For the parabola $y = (x - 4)^2 - 5$ write down:
   (i) The co-ordinates of the vertex. (1 mark)
   (ii) The minimum value of the function. (1 mark)
   (iii) The co-ordinates of the focus. (1 mark)

3. Write down the equation of the directrix of the parabola: $x^2 = -8y$. (1 mark)

4. Find the vertex and focus of $(x+2)^2 = -12y + 6$. (2 marks)

5. Given the parabola $12y = x^2 - 4x + 52$ express it in the form $(x - x_1)^2 = 4a(y - y_1)$. Find:
   (i) The co-ordinates of the vertex. (1 mark)
   (ii) The focal length. (1 mark)
   (iii) The co-ordinates of the focus. (1 mark)
   (iv) The equation of the directrix. (1 mark)
   (v) The equation of the axis of the parabola. (1 mark)

6. For the parabola $y = \frac{1}{8}x^2 - x + 3$
   (i) Find the co-ordinates of the vertex and focus. (2 marks)
   (ii) Find the equation of the normal to the parabola at the point where $x = -4$. Write your answer in general form. (1 mark)
   (iii) Find the point on the parabola at which the tangent is parallel to $y = 3x + 1$. (1 mark)

7. Consider the parabola $x^2 = 4(y-5)$
   (i) Write down the co-ordinates of the vertex. (1 mark)
   (ii) What are the co-ordinates of the focus. (1 mark)
   (iii) Sketch the parabola $x^2 = 4(y-5)$. (1 mark)

8. Given the parabola $(x+2)^2 = 8(y-1)$ write down:
   (i) The co-ordinates of the focus. (1 mark)
   (ii) The equation of the directrix. (1 mark)

9. A parabola has the equation $x^2 = 12(8-y)$. What is the equation of the directrix? (1 mark)

10. The parabola P has equation $x^2 = 16(y+1)$

(i) Draw a neat $\frac{1}{3}$ page sketch of P and clearly indicate on it:

a) The equation of the directrix. (1 mark)

b) The co-ordinates of its focus. (1 mark)

c) The co-ordinates of its vertex. (1 mark)

(ii) Find the co-ordinates of the point where P cuts the x axis. (1 mark)

11. Sketch the locus of the point $P(x, y)$ which moves so that it is equidistant from the point $(1,2)$ and the line $y = -2$. Describe the locus of P geometrically and find its equation. (3 marks)

12. The equation $(x-2)^2 = 6\left(y - \frac{1}{2}\right)$ represents a parabola.
    - (i) Write down the co-ordinates of the vertex. (1 mark)
    - (ii) Find the focal length. (1 mark)
    - (iii) Find the x intercept of the parabola. (1 mark)
    - (iv) Sketch the graph of the parabola, show the focus and directrix on your diagram. (3 marks)

End of homework