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2018 HIGHER SCHOOL CERTIFICATE COURSE MATERIALS

HSC Mathematics Extension 1

Complex Numbers

Term 1 – Week 3

Name ……………………………………………………………….

Class day and time …………………………………………………

Teacher name ………………………………………………...

Term 1 – Week 3 – Theory

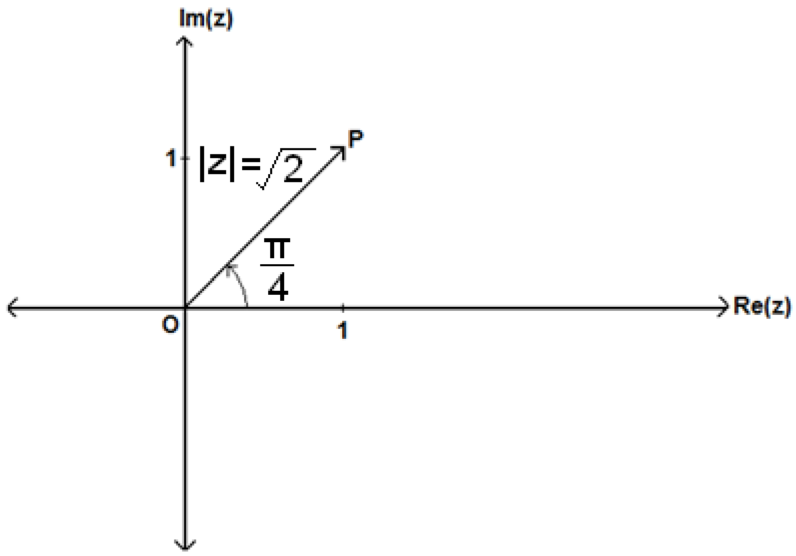
GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER AS A VECTOR:

REPRESENTING A COMPLEX NUMBER AS A VECTOR ON AN ARGAND DIAGRAM:

Vectors are quantities which have both magnitude and direction, and thus can be represented by arrows in the direction of the quantity. The length of the arrow indicates the magnitude of the vector.

Complex Numbers can be represented as vectors on an Argand diagram, similar to the way they can be represented as points. The only difference now is that an arrow extends from the origin to the point with the tip pointing at the Complex Numbers geometrical point. The magnitude of the vector in this case is the modulus of the Complex Number, and the argument of the Complex Number indicates the direction.

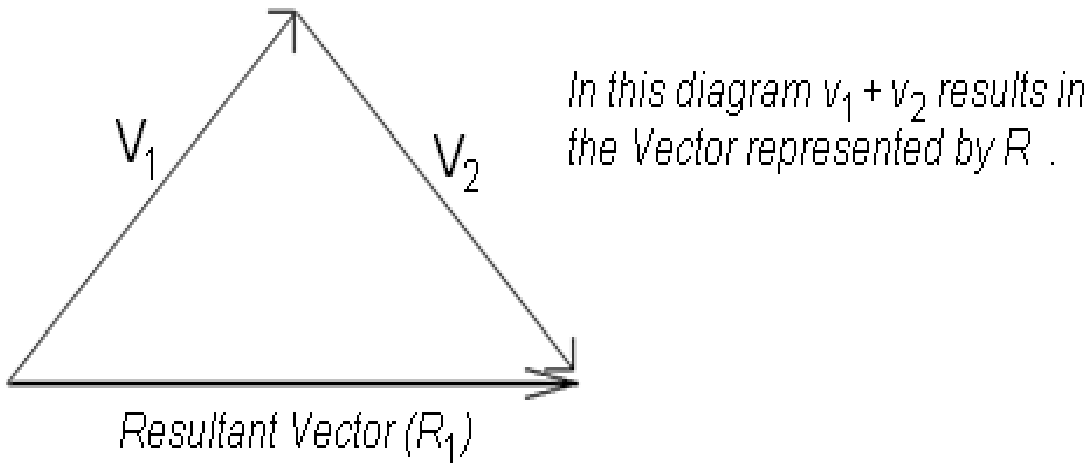
For example in the diagram below the Complex Number $1+i$ is represented by the vector $\overrightarrow{\mathrm{OP}}$ where P is the point $(1,1)$. Note that the arrow indicates the direction and sign of the Complex Number. By putting the arrow in the opposite direction $\left(\overleftarrow{\mathrm{OP}}\right)$ we indicate that the vector is in the opposite direction, and that the sign of the Complex Number is opposite. This indication of an arrow above the interval representing the Complex Number is due to the fact that Vectors can be *shifted* or *translated.* This will be further dealt with later. Now, the Complex Number
$1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ in modulus argument form. Thus the direction is $\frac{\pi}{4}$ anti-clockwise rotation from the positive real axis, and the magnitude is $\sqrt{2}$.



GEOMETRICAL REPRESENTATION OF THE ADDITION AND SUBTRACTION OF TWO COMPLEX NUMBERS:

When two Complex Numbers are added or subtracted as well as being able to analyse this arithmetically, we can also analyse it geometrically, using vector representation.

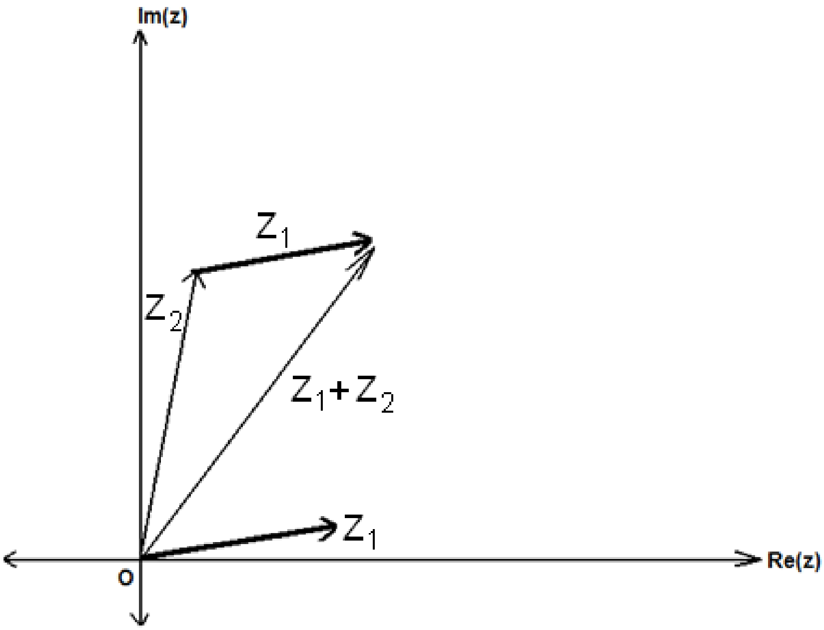
When adding or subtracting vectors, the vectors must be aligned head to tail. i.e. the vectors must be arranged so that the tip of one vector is at the end of the other. The resultant vector is the vector obtained by connecting the first tail with the last head. The diagram illustrates this concept.



Now that we know how to add Vectors the question “How do we add Complex Numbers?” arises.

As mentioned earlier, due to the fact that most Complex Numbers will have their Vectors directed from the origin, it becomes impossible to add vectors by leaving them as they are. We must somehow align them from tip to tail. This can be achieved by *Shifting* or *Translating* the Complex Number from its original position, to a position where the tail is aligned with the tip of another arrow which we wish to add it to. This way we can draw in the resultant Vector. It should be added that by translating the Vector, NO change has occurred in the vector itself, since its argument (angle with positive axis) and modulus remain the same. Thus this means that the translated and original vector are parallel, and this can be seen in the diagram.

The diagram below illustrates these concepts.

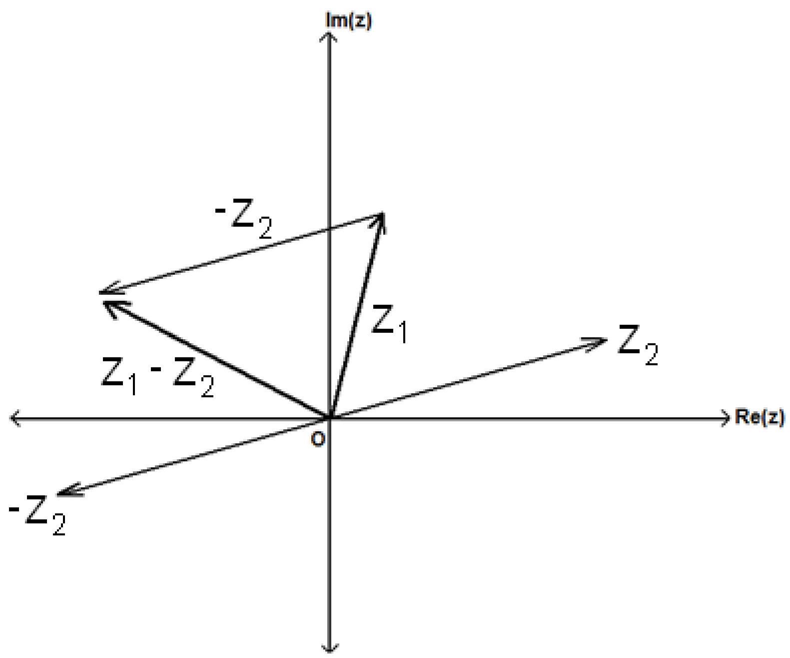


The Subtraction of Complex Numbers using vector representation, uses the same process except that the vector being subtracted must firstly be redirected *then* translated. i.e. $z_1 - z_2$ must be treated the same as $z_1 + (-z_2)$.

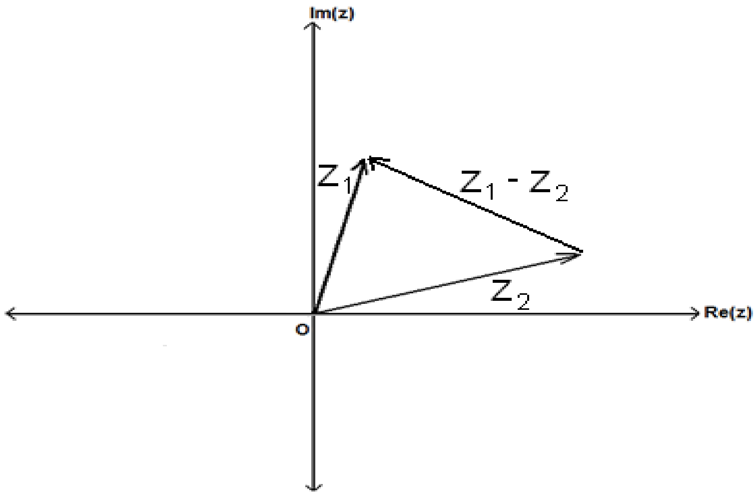
Firstly, the transformation from z_2 to $-z_2$ involves a π radians rotation of the vector representing z_2.

Then the Complex Number must be translated, and only then can the resultant vector be obtained.

The diagram below illustrates this process.



Although the resultant vector $z_1 - z_2$ is correctly drawn in the diagram above, it is usual to draw the vector $z_1 - z_2$ as shown in the diagram below,

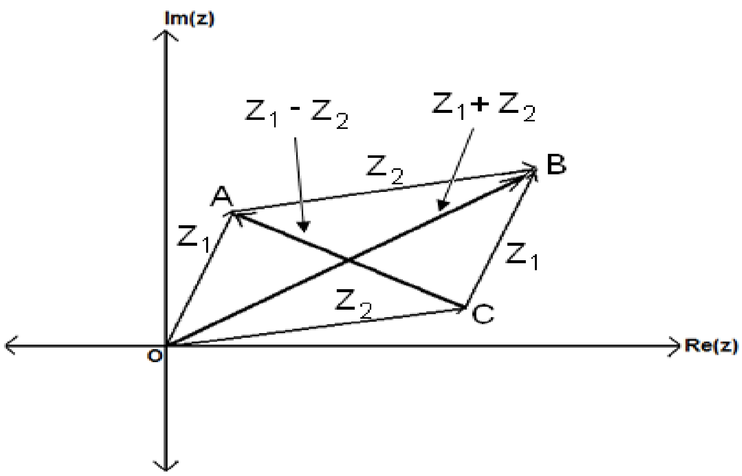


The reason for this representation is due to the vector $z_1 - z_2$ being the diagonal of a triangle and parallelogram on the Argand diagram. This will be studied in detail in the next section.

GEOMETRICAL SIGNIFICANCE OF $z_1 + z_2$, $z_1 - z_2$ GIVEN z_1 , z_2 ON THE ARGAND DIAGRAM:

We have so far only studied *how* to geometrically add or subtract two vectors on an Argand diagram, in this section we intend to study the significance of $z_1 + z_2$, $z_1 - z_2$ when considered on an Argand diagram.

It is obvious that both $z_1 + z_2$, $z_1 - z_2$ are sides of a triangle when considered on the Argand diagram. This can be seen from the diagrams in the previous section. However, what may not be so obvious is that they correspond to the diagonals of a parallelogram as well, after the translation of vectors. The diagram below shows this concept.



It can be seen from the diagram that the vectors $\overrightarrow{OB}$ and $\overrightarrow{CA}$ which represent $z_1 + z_2$, $z_1 - z_2$, are diagonals of the parallelogram OABC.

Now that we know that $z_1 + z_2$, $z_1 - z_2$ correspond to diagonals on a parallelogram, we shall study the properties of these Complex Numbers given certain conditions for z_1 and z_2. It must be noted that the geometrical properties of parallelograms and triangles must be used to answer questions which involve vectors and their geometrical properties.

1. Given $|z_1| = |z_2|$.

In this case if $|z_1| = |z_2|$, then all four sides of the parallelogram are equal, and thus the parallelogram is in fact a rhombus.

The main properties of the rhombus that are used in HSC Mathematics Extension 2, is the fact that the <u>diagonals are perpendicular to each other</u>, and that the <u>diagonals bisect the interior angles through which they pass</u>.

This means that $z_1 - z_2 = ki(z_1 + z_2)$ where $k \in R$, due to the diagonals being perpendicular.

It also means there are certain relations between the arguments of z_1, z_2, $z_1 + z_2$, $z_1 - z_2$, due to the diagonals bisecting the interior angles through which they pass through.

2. Given $z_1 = kiz_2$ *where* $k \in R$, $k \neq \pm 1$.

In this case, $|z_1| \neq |z_2|$ but there is a right angle between z_1 and z_2, thus implying that the parallelogram formed is in fact a rectangle.

The properties of rectangles which are primarily used in this topic are the <u>right angle present between adjacent sides and the equal length of the diagonals</u>.

The latter property implies the relationship, $|z_1 + z_2| = |z_1 - z_2|$.

3. Given $z_1 = iz_2$.

In this case both $|z_1| = |z_2|$, and the adjacent sides are perpendicular. Thus, the parallelogram formed is in fact a square.

The main properties used in this case are that the <u>diagonals are equal in length and perpendicular</u>, and that the <u>diagonals bisect the interior angles</u>.

The first property implies that $z_1 - z_2 = i(z_1 + z_2)$.

GIVEN z_1 AND z_2, CONSTRUCT THE VECTOR z_1z_2:

We have studied the multiplication of two Complex Numbers when the Complex Numbers are in the form $x + iy$ and when the Complex Numbers are in the form $r\ cis\ \theta$. We can obviously graph the Complex Number z_1z_2 when z_1 and z_2 are in the form $x + iy$ by firstly carrying out the binomial expansion, then graphing the resulting vector. We can also graph the Complex Number z_1z_2 if z_1 and z_2 are each given in mod – arg form, by noting that the arguments will add together and that the moduli will multiply together to give the resultant product. Then we can simply graph the resulting Complex Number as a vector on an Argand diagram.

The following example illustrates this technique.

EXAMPLE:

Graph the Complex Number zw on the Argand Plane as a vector, given that $z = 2\ cis\frac{\pi}{6}$ and $w = 4\ cis\frac{\pi}{12}$.

SOLUTION:

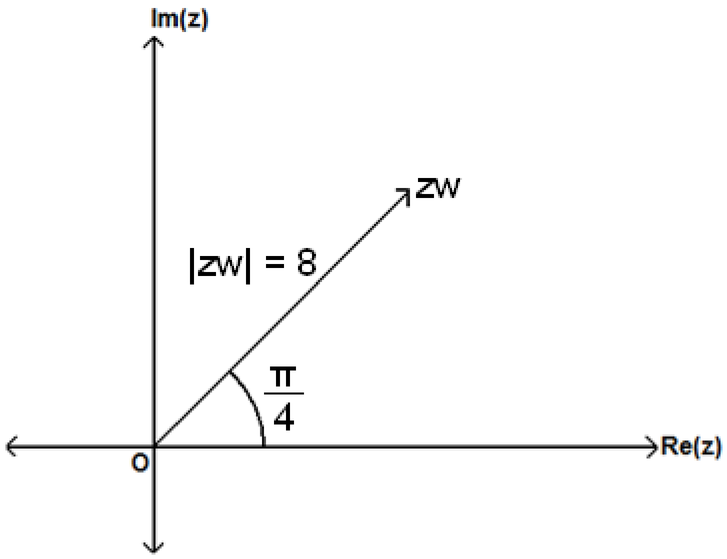
Firstly we find zw.

$$zw = 2\ cis\ \frac{\pi}{6} \times 4\ cis\ \frac{\pi}{12}$$

$$= 8\ cis\ \left(\frac{\pi}{6} + \frac{\pi}{12}\right)$$

$$= 8\ cis\ \frac{\pi}{4}$$

Now we graph the vector on the Argand diagram.



Whenever drawing the product of two vectors, regardless of whether the constituent vectors are in mod – arg form, or if they are graphed for you, it is vital to remember that the vector of the product can be simply drawn by adding the arguments and then by multiplying the moduli.

THE TRIANGULAR INEQUALITIES:

From these geometrical representations of Complex Numbers as vectors, many different relationships can arise. One of interest is the triangular inequality, which simply states,

$$|a| + |b| \geq |a + b| \text{ where } a, b \in C$$

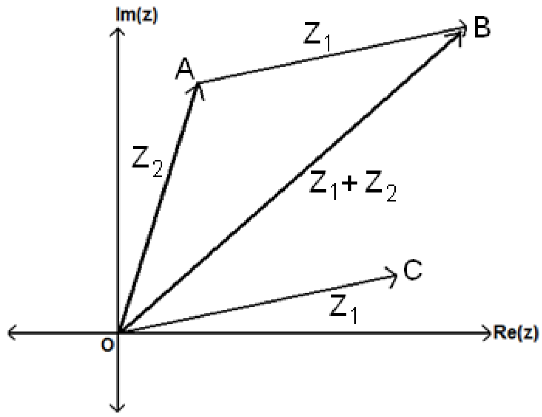
The proof is given in the example below.

EXAMPLE:

Prove the triangular inequality, $|z_1| + |z_2| \geq |z_1 + z_2|$.

SOLUTION:

Firstly we will draw a graph of two arbitrary Complex Numbers z_1, z_2 and their sum.



From the graph it is obvious that BAO is a triangle.

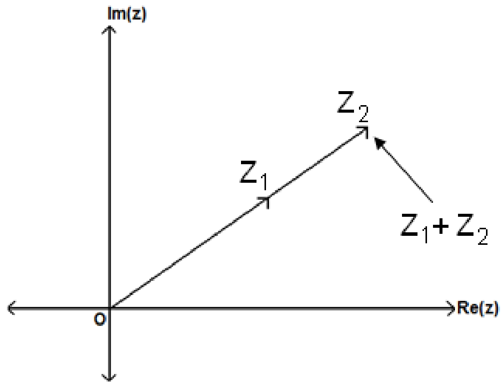
Now, we know that the sum of two sides of a triangle is always greater than the other two sides.

So, $AO + AB > OB$.

i.e. $|z_2| + |z_1| > |z_1 + z_2|$

We need to establish whether equality ever holds.

Consider the case when z_1 and z_2 are collinear.



It is obvious from this case that $|z_1| + |z_2| = |z_1 + z_2|$, when z_1 , z_2 are collinear.

Thus $|z_1| + |z_2| \geq |z_1 + z_2|$.

It should be obvious after the above example why it is called the triangular inequality.

Term 1 – Week 3 – Homework

1. Represent the following Complex Numbers as vectors on an Argand diagram.
   a) i
   b) $1 + i$
   c) $-1 + i$
   d) $2\ cis\ \frac{\pi}{6}$
   e) $3\ cis\left(-\frac{5\pi}{6}\right)$
   f) $2\ cis\ \frac{\pi}{3} \times \frac{1}{2}\ cis\ \frac{\pi}{6}$
   g) $(1 + i)\left(1 + \sqrt{3}i\right)$
   h) $\frac{2\ cis\frac{\pi}{4}}{3\ cis\frac{\pi}{12}}$

2. Given $z_1 = 2\ cis\ \frac{\pi}{6}$ and $z_2 = 2\ cis\ \frac{\pi}{3}$, sketch on an Argand plane,
   a) $z_1 z_2$
   b) $\frac{z_2}{z_1}$
   c) $z_1 + z_2$
   d) $z_2 - z_1$
   e) ${z_1}^2$
   f) ${z_2}^3$

3. Show on an Argand diagram the vectors represented by z, iz, $\frac{1}{z}$, clearly showing the relationship between their mods and args.

4. A parallelogram ABCD is drawn with $\overrightarrow{OA} = 3 + 2i$, $\overrightarrow{OC} = 11 + 9i$ and $\overrightarrow{AB} = 2 + 5i$. Find the vectors $\overrightarrow{OB}$ and $\overrightarrow{OD}$. *(Hint:* $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$*)*

5. On an Argand diagram the parallelogram OPQR where O is the origin, has a vertex Q as the point representing the Complex Number $z_1 = 3 + 2i$ and the diagonal $\overrightarrow{PR}$ is represented by a vector $z_2 = -10 + 15i$.

a) Find the coordinates of the vertices P and R.

b) Evaluate $\frac{z_2}{z_1}$ in the form $a + ib$.

c) Describe the relationship between z_2 and z_1, and define the type of parallelogram.

6. The four Complex Numbers z_1 , z_2 , z_3 , z_4 are represented on the Argand Plane by the points A, B, C, D respectively. If $z_1 - z_2 + z_3 - z_4 = 0$ and $z_1 - iz_2 - z_3 + iz_4 = 0$, determine the possible shape (s) for the quadrilateral ABCD.

7. Show by induction that $|z_1 + z_2 + z_3 + z_4 + \cdots + z_n| \leq |z_1| + |z_2| + |z_3| + \cdots + |z_n|$ assuming that $|z_1 + z_2| \leq |z_1| + |z_2|$.

8. P, Q and R represent the Complex Numbers z_1, z_2, z_3 respectively, where $z_1 z_2 = {z_3}^2$. Show geometrically that OR bisects the angle POQ.

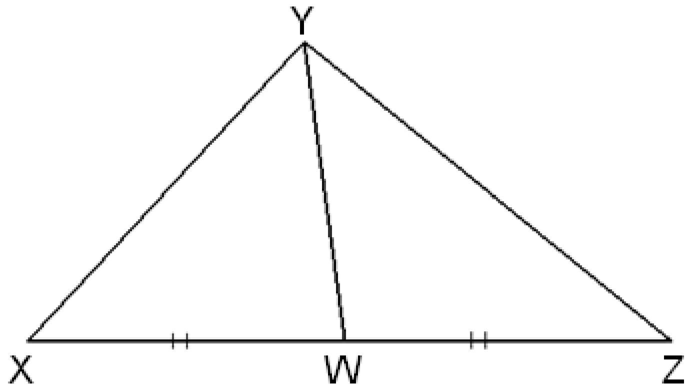
9.

a) Prove $|z|^2 = z\bar{z}$

b) A parallelogram OABC has diagonals intersecting at M. Let OA be z_1, OC be z_2 and $0 < \arg z_1 < \arg z_2 < \frac{\pi}{2}$. Draw the parallelogram OABC clearly showing the point M and the vectors $z_1,\ z_2,\ z_1 + z_2,\ z_1 - z_2$.

c) Prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$.

d) Hence prove Apollonius' Theorem for a triangle XYZ, $XY^2 + YZ^2 = 2(YW^2 + XW^2)$.



10. In the Argand diagram, the points Z_1, Z_2 represent the Complex Numbers z_1 , z_2 respectively. Show that if the triangle OZ_1Z_2 is isosceles and right angled at O, then ${z_1}^2 + {z_2}^2 = 0$.

11. If $\alpha, \beta, \gamma, \delta, \mu, \omega$ are six Complex Numbers and $\frac{\alpha-\gamma}{\alpha-\beta} = \frac{\delta-\omega}{\delta-\mu}$ show that the triangles represented by these numbers are similar.

12. If α, β, γ are 3 Complex Numbers represented by the points A, B, C on the Argand diagram and $\frac{\alpha-\gamma}{\beta-\gamma} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ prove that the triangle ABC is equilateral, and that $\alpha^2 + \beta^2 + \gamma^2 = \alpha\beta + \beta\gamma + \alpha\gamma$. If $\alpha = 1$, $\beta = i$ find the two possible positions for γ.

13. P, Q, R, S is a parallelogram. H is the point of intersection of the diagonals. If the points P, R, S represent Complex Numbers $1 + 3i$, $2 + 6i$, $5 + 7i$ respectively, find the Complex Numbers giving the points Q, H.

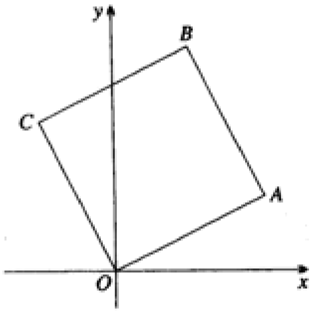
14. E is the centre of a square ABCD, lettered anti-clockwise. E, A are the points $-2+i$, $1+5i$ respectively. Find the Complex Numbers giving the vertices B, C, D.

15. If the Complex Numbers z_1 , z_2 , z_3 are represented on the Argand diagram by the points P, Q, R respectively and the angles of ΔPQR at Q, R are each $\frac{1}{2}(\pi-\alpha)$, prove that $(z_3-z_2)^2 = 4(z_3-z_1)(z_1-z_2)\sin^2\left(\frac{\alpha}{2}\right)$.

16. In the Complex Plane the points $P_1,\ P_2$ and P_3 represent the Complex Numbers $z_1,\ z_2,\ z_3$ respectively. If P_1', P_2' and P_3' represent the numbers $z_2 + z_3,\ z_3 + z_1,\ z_1 + z_2$ respectively, show that the triangles $P_1P_2P_3$ and $P_1'P_2'P_3'$ are congruent.

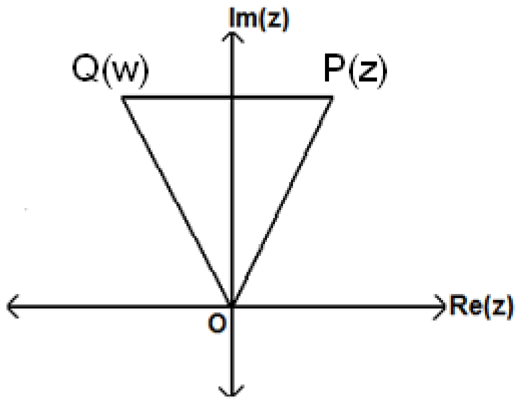
17. The origin O and the points A, B and C representing the complex numbers $z, \frac{1}{z}$ and $z+\frac{1}{z}$ respectively are joined to form a quadrilateral. Write down the condition or conditions for z so that the quadrilateral OABC will be
    a) a rhombus
    b) a square

18.



In the square OABC shown above, the point A represents $2+i$. What Complex Numbers do the points B and C represent?

19. Points P and Q represent the complex numbers z and w respectively in the Argand diagram below. If ΔOPQ where O is the origin, is equilateral,
    a) Explain why $wz = z^2 cis\frac{\pi}{3}$.
    b) Prove that $z^2 + w^2 = zw$



End of Homework