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2018 HIGHER SCHOOL CERTIFICATE COURSE MATERIALS

HSC Mathematics Extension 1

Projectile Motion II

Term 1 – Week 7

Name ……………………………………………………….

Class day and time ……………………………………………

Teacher name ………………………………………………...

Term 1 – Week 7 – Theory

PROJECTILE MOTION:

EXAMPLE:

A soccer player kicks a ball on the ground with initial velocity $30\ \mathrm{m/s}$ at an angle of $30°$. Assuming the acceleration due to gravity is $10\ \mathrm{m/s^2}$. Find:

(i) The Cartesian equation of the trajectory.
(ii) The time of flight and the range.
(iii) The maximum height and the time when it occurs.
(iv) The speed and angle when the ball strikes the ground.

SOLUTION:

(i)

Horizontal Motion

$$\therefore \ddot{x} = 0$$

$$\dot{x} = \int 0\,.dt$$

$$\dot{x} = C_1$$

When $t = 0$, $\dot{x} = V\cos\alpha$

$$\dot{x} = 30\cos 30°$$

$$\dot{x} = 15\sqrt{3}$$

$$\therefore \dot{x} = 15\sqrt{3}$$

$$x = \int 15\sqrt{3}\,.dt$$

$$x = 15\sqrt{3}t + C_2$$

When $t = 0, x = 0$,

$$0 = 15\sqrt{3}(0) + C_2$$

$$C_2 = 0$$

$$\therefore x = 15\sqrt{3}t$$

Vertical Motion

$$\therefore \ddot{y} = -10$$

$$\dot{y} = \int -10\,.dt$$

$$\dot{y} = -10t + D_1$$

When $t = 0$, $\dot{y} = V\sin\alpha$

$$\dot{y} = 30\sin 30°$$

$$\dot{y} = 15$$

$$15 = -10(0) + D_1$$

$$D_1 = 15$$

$$\therefore \dot{y} = 15 - 10t$$

$$y = \int 15 - 10t\,.dt$$

$$y = 15t - 5t^2 + D_2$$

When $t = 0, y = 0$,

$$0 = 15(0) - 5(0)^2 + D_2$$

$$D_2 = 0$$

$$\therefore y = 15t - 5t^2$$

Substitute $t = \frac{x}{15\sqrt{3}}$ into $y = 15t - 5t^2$,

$$y = 15\left(\frac{x}{15\sqrt{3}}\right) - 5\left(\frac{x}{15\sqrt{3}}\right)^2$$
$$= \frac{x}{\sqrt{3}} - 5\left(\frac{x^2}{675}\right)$$
$$\therefore y = \frac{x}{\sqrt{3}} - \frac{x^2}{135},$$ which is the Cartesian equation of the trajectory

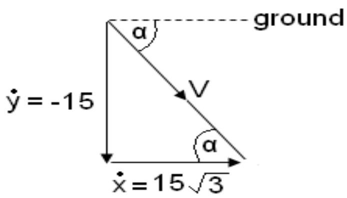
(ii) When $y = 0$,
$15t - 5t^2 = 0$
$5t(3 - t) = 0$
$t = 0, \quad 3$
Since $t = 0$ is the starting point so $t = 3$ is the time of flight.

When $t = 3$,
$x = 15\sqrt{3}t$
$x = 15\sqrt{3}(3)$
$\therefore x = 45\sqrt{3}$ m
∴ The range is $45\sqrt{3}$ metres.

(iii) When $\dot{y} = 0$,
$15 - 10t = 0$
$10t = 15$
$t = 1.5$ seconds

When $t = 1.5$,
$y = 15t - 5t^2$
$y = 15(1.5) - 5(1.5)^2$
$y = 11.25$ m
∴ The maximum height is 11.25 metres and it occurs after 1.5 seconds.

(iv) When $t = 3$,
$\dot{x} = 15\sqrt{3}$ and
$\dot{y} = 15 - 10t$
$\dot{y} = 15 - 10(3)$
$\dot{y} = -15$



By Pythagoras' Theorem,

$$V_{final} = \sqrt{(-15)^2 + \left(15\sqrt{3}\right)^2}$$

$$\therefore V_{final} = 30 \text{ m/s}$$

$$\tan\alpha = -\frac{15}{15\sqrt{3}}$$

$$\tan\alpha = -\frac{1}{\sqrt{3}}$$

$$\therefore \alpha = -30°$$

∴ The speed is 30 m/s and the angle is $-30°$ when the ball strikes the ground.

EXAMPLE:

A tennis ball is thrown from the top of a $30\ \mathrm{m}$ building at an angle α to the ground level where $\tan\alpha = \frac{4}{3}$ with an initial velocity of $20\ \mathrm{m/s}$. Neglecting air resistance and assuming the acceleration due to gravity is $10\ \mathrm{m/s^2}$, find:

(i) The Cartesian equation of the path.
(ii) The maximum height reached and the time when it occurs.

SOLUTION:

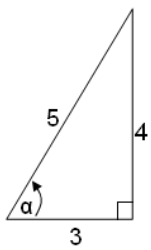
(i)

Horizontal Motion

$$\therefore \ddot{x} = 0$$

$$\dot{x} = \int 0\,.\,dt$$
$$\dot{x} = C_1$$

When $t = 0$, $\dot{x} = V\cos\alpha$



$$\dot{x} = 20\left(\frac{3}{5}\right)$$
$$\dot{x} = 12$$

$$\therefore \dot{x} = 12$$

$$x = \int 12\,.\,dt$$
$$x = 12t + C_2$$

When $t = 0, x = 0$,

$$0 = 12(0) + C_2$$
$$C_2 = 0$$

$$\therefore x = 12t$$

Vertical Motion

$$\therefore \ddot{y} = -10$$

$$\dot{y} = \int -10\,.\,dt$$
$$\dot{y} = -10t + D_1$$

When $t = 0$, $\dot{y} = V\sin\alpha$

$$\dot{y} = 20\left(\frac{4}{5}\right)$$
$$\dot{y} = 16$$

$$16 = -10(0) + D_1$$
$$D_1 = 16$$

$$\therefore \dot{y} = 16 - 10t$$

$$y = \int 16 - 10t\,.\,dt$$
$$y = 16t - 5t^2 + D_2$$

When $t = 0, y = 30$,

$$30 = 15(0) - 5(0)^2 + D_2$$
$$D_2 = 30$$

$$\therefore y = 30 + 16t - 5t^2$$

Substitute $t = \frac{x}{12}$ into $y = 30 + 16t - 5t^2$,

$$y = 30 + 16\left(\frac{x}{12}\right) - 5\left(\frac{x}{12}\right)^2$$
$$\therefore y = 30 + \frac{4}{3}x - \frac{5}{144}x^2$$

(ii) When $\dot{y} = 0$,

$16 - 10t = 0$

$10t = 16$

$t = 1.6$ seconds

When $t = 1.6$,

$y = 30 + 16(1.6) - 5(1.6)^2$

$y = 42.8$ metres

$\therefore$ The maximum height is 42.8 metres and it occurs after 1.6 seconds.

EXAMPLE:

A ball is thrown so that it just clears a wall that is 25 metres horizontally and 13 metres vertically from the point of projection. If its range is 45 metres, find the speed and angle of projection.

SOLUTION:

Horizontal Motion

$$\therefore \ddot{x} = 0$$

$$\dot{x} = \int 0\,.\,dt$$
$$\dot{x} = C_1$$

When $t = 0, \dot{x} = V\cos\alpha$,
$$C_1 = V\cos\alpha$$

$$\therefore \dot{x} = V\cos\alpha$$

$$x = \int V\cos\alpha\,.\,dt$$
$$x = V\cos\alpha\, t + C_2$$

When $t = 0, x = 0$,

$$0 = V\cos\alpha\,(0) + C_2$$
$$C_2 = 0$$

$$\therefore x = V\cos\alpha\, t$$

Vertical Motion

$$\therefore \ddot{y} = -10$$

$$\dot{y} = \int -10\,.\,dt$$
$$\dot{y} = -10t + D_1$$

When $t = 0, \dot{y} = V\sin\alpha$,

$$V\sin\alpha = -10(0) + D_1$$
$$D_1 = V\sin\alpha$$

$$\therefore \dot{y} = V\sin\alpha - 10t$$

$$y = \int V\sin\alpha - 10t\,.\,dt$$
$$y = V\sin\alpha\, t - 5t^2 + D_2$$

When $t = 0, y = 0$,

$$0 = V\sin\alpha\,(0) - 5(0)^2 + D_2$$
$$D_2 = 0$$

$$\therefore y = V\sin\alpha\, t - 5t^2$$

The Cartesian equation of the trajectory can be found by substituting $t = \frac{x}{V\cos\alpha}$ into $y = V\sin\alpha\, t - 5t^2$.

$$y = V\sin\alpha\left(\frac{x}{V\cos\alpha}\right) - 5\left(\frac{x}{V\cos\alpha}\right)^2$$

$$y = x\tan\alpha - \frac{5x^2}{V^2}\sec^2\alpha$$

Given that when $x = 45, y = 0$ and $x = 25, y = 13$,

So,

$$45\tan\alpha - \frac{5(45)^2}{V^2}\sec^2\alpha = 0$$

$$45\tan\alpha - \frac{10125}{V^2}\sec^2\alpha = 0$$

$$\frac{10125\sec^2\alpha}{V^2} = 45\tan\alpha$$

$$V^2 = \frac{10125\sec^2\alpha}{45\tan\alpha}$$

$$V^2 = \frac{225\sec^2\alpha}{\tan\alpha} ----(1)$$

And,

$$13 = 25\tan\alpha - \frac{5(25)^2}{V^2}\sec^2\alpha$$

$$13 = 25\tan\alpha - \frac{3125\sec^2\alpha}{V^2}$$

$$\frac{3125\sec^2\alpha}{V^2} = 25\tan\alpha - 13$$

$$V^2 = \frac{3125\sec^2\alpha}{25\tan\alpha - 13} ----(2)$$

Equate equations (1) and (2),

$$\frac{225\sec^2\alpha}{\tan\alpha} = \frac{3125\sec^2\alpha}{25\tan\alpha - 13}$$

$$225(25\tan\alpha - 13) = 3125\tan\alpha$$

$$5625\tan\alpha - 2925 = 3125\tan\alpha$$

$$2500\tan\alpha = 2925$$

$$\tan\alpha = \frac{117}{100}$$

$$\therefore \alpha \approx 49°29'$$

When $\alpha = 49°29'$,

$$V^2 = \frac{225\sec^2\alpha}{\tan\alpha}$$

$$= \frac{225\sec^2(49°29')}{\tan(49°29')}$$

$$= 455\frac{29}{52}$$

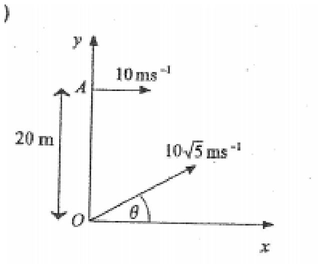
$$\therefore V \approx 21.34 \text{ m/s}$$

Term 1 – Week 7 – Homework

Projectile Motion:

1. A projectile at the highest point of its trajectory has a velocity 8 metres per second and its position is 8 metres above the ground. Find, taking $g = 9.8\text{ms}^{-2}$. (5 marks)
   (i) The angle of projection (to the nearest degree).
   (ii) The initial velocity (correct to 1 decimal place).

2. OA is a vertical building of height 20 metres. A particle is projected horizontally from A with speed 10m/s. At the same instant, a second particle is projected from O with speed $10\sqrt{5}$m/s at an angle θ above the horizontal. The two particles travel in the same plane of motion. Take $g = 10$m/s^2.

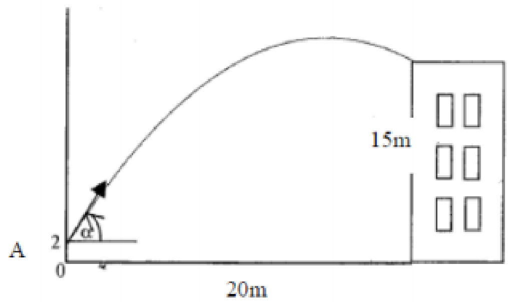


(i) Write down expressions for horizontal and vertical displacements relative to O for each particle after time t seconds. (2 marks)

(ii) Show that if the two particles collide, then they do so after 1 second. (2 marks)

(iii) Show that if the two particles collide, when they do so their paths of motion are perpendicular to each other. (2 marks)

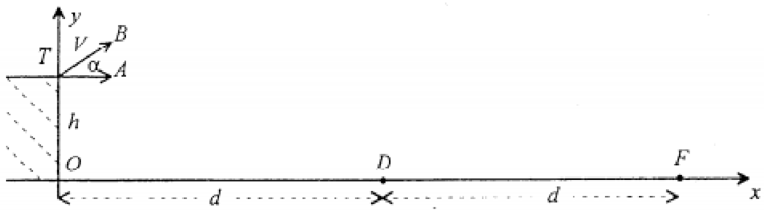
3. Andrew, whose height is 2 metres, throws a ball from area A in the direction of the Cohen building which is 15 metres high. He throws the ball with an initial velocity u at angle α, and he is 20 metres from the base of the building. Assume that $\ddot{x} = 0$ and $\ddot{y} = -10\text{m/s}^2$.



(i) Show that $y = x\tan\alpha - \frac{5x^2}{u^2}(1 + \tan^2\alpha) + 2$ at any time t. (2 marks)

(ii) Hence, find between which two angles of projection must he throw the ball to ensure that it lands on the roof of the building, or over, given that $u = 25$m/s, to the nearest degrees. (3 marks)

4. T is the top of a building h metres high. The points O, D and F are in the same line on flat level ground. O is the base of the building. D is d metres from O, and F is a further d metres from D. At time $t = 0$, two particles A and B are projected with the same initial velocity Vm/s from T. Particle A is projected horizontally and particle B is projected in the same direction, but at an angle α, $\alpha > 0$, to the horizontal. The equations of motion are $\ddot{x} = 0$ and $\ddot{y} = -g$.



(i) Assuming that the position of particle A at time t is given by
$x = Vt, y = -\frac{1}{2}gt^2 + h$
Show that the Cartesian equation of the trajectory is given by $y = h - \frac{g}{2V^2}x^2$. (1 mark)

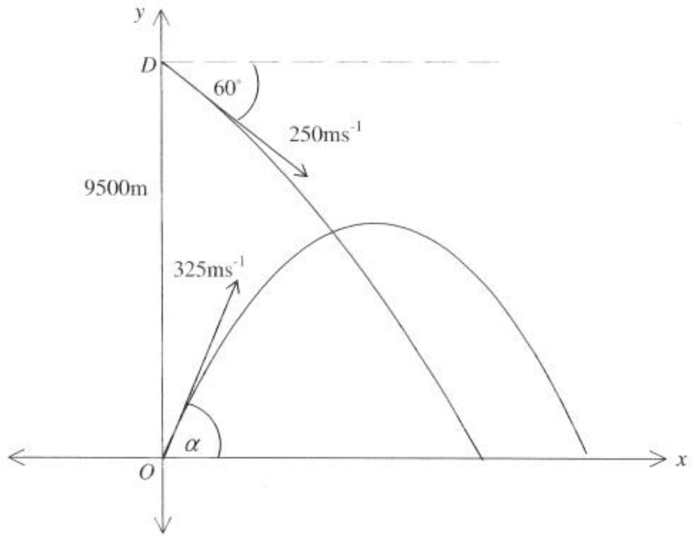
(ii) Assuming that the position of particle B at time t is given by $x = Vt\cos\alpha$ and $y = -\frac{1}{2}gt^2 + Vt\sin\alpha + h$. Show that the Cartesian equation of the trajectory is given by $y = x\tan\alpha - \frac{gx^2}{2V^2}\sec^2\alpha + h$. (1 mark)

(iii) If A lands at D show that $h = \frac{gd^2}{2V^2}$. (1 mark)

(iv) If both A and B land at D show that $\tan\alpha = \frac{d}{h}$. (2 marks)

(v) If A lands at D and B lands at F show that $d \geq 2h\sqrt{3}$. (3 marks)

5. During an army exercise, a surface to air missile is launched from the point O in order to intercept a dummy bomb that is released from a point D. The point D is 9500 metres directly above O.



The dummy bomb is released at an angle of 60° below the horizontal, with a velocity of 250m/s. It can be shown that the equations of motion of the dummy bomb are:

$x_D = 125t$ and $y_D = 9500 - 125\sqrt{3}t - 5t^2$. (Do NOT prove this)

(i) Calculate how long it would take the dummy bomb to reach the ground (correct to the nearest second) and where it would strike the ground (correct to the nearest metre). (2 marks)

The missile is launched at the same time as the dummy bomb is released. It is launched with an initial velocity of 325m/s and its angle of projection above the horizontal is α. The equations of motion of the missile are:

$x_M = 325t\cos\alpha$ and $y_M = 325t\sin\alpha - 5t^2$. (Do NOT prove this)

(ii) Show that in order for the missile to intercept the dummy bomb it must be launched with an angle of projectile $\alpha = \cos^{-1}\left(\frac{5}{13}\right)$. (1 mark)

(iii) How high above the ground, correct to the nearest metre, does the collision occur? (3 marks)

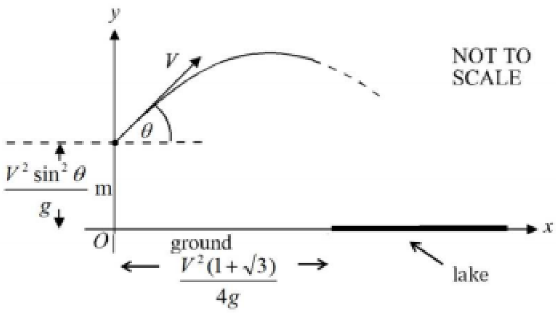
6. Two flares are fired from a boat at the same time and with the same velocity V. Flare A is fired vertically upwards. Flare B is fired at an angle of inclination to the horizontal of θ.

(i) Ignoring air resistance, show that the height of flare A is: $y_A = -\frac{1}{2}gt^2 + Vt$. (2 marks)

(ii) Show that the difference between the maximum height of the two flares is $\frac{V^2}{2g}\cos^2\theta$. You may assume that the height of flare B is $y_B = -\frac{1}{2}gt^2 + Vt\sin\theta$. (3 marks)

(iii) It is known that the best chance for the two flares to be seen is for the ratio of the range of flare B, to the difference in maximum height of the two flares, to be $4 : \sqrt{3}$. Find the value of θ that gives the best chance of the two flares being seen. (3 marks)

7. An archer stands at the edge of a cliff and shoots an arrow at a constant velocity of Vm/s and at an angle θ to the horizontal, where $0 < \theta < 90°$. The arrow that he shoots is released from a point $\frac{V^2 \sin^2 \theta}{g}$m vertically above the ground. At ground level, $\frac{V^2(1+\sqrt{3})}{4g}$m away horizontally from the point of projection is a lake that is $\frac{V^2}{2g}$m wide. The position of the arrow at time t seconds after it is projected is given by:
$x = Vt\cos\theta$ and $y = -\frac{gt^2}{2} + Vt\sin\theta + \frac{V^2\sin^2\theta}{g}$. (1 mark)

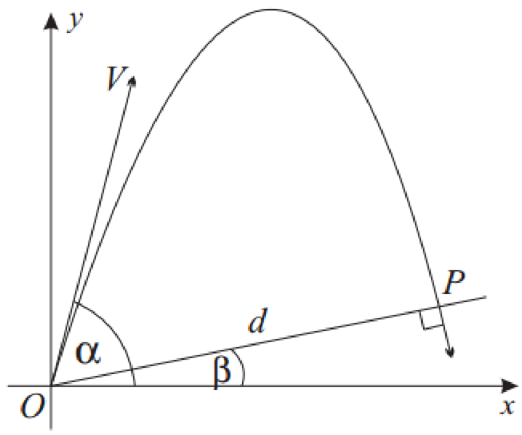


(i) Show that the Cartesian equation of the path of the arrow is given by:
$y = -\frac{gx^2}{2V^2}\sec^2\theta + x\tan\theta + \frac{V^2\sin^2\theta}{g}$. (1 mark)

(ii) Show that the horizontal range of the arrow on the ground is given by $x = \frac{V^2(1+\sqrt{3})\sin 2\theta}{2g}$. (2 marks)

(iii) Find the values of θ for which the arrow will not land in the lake or on the edge of the lake. (4 marks)

8. The diagram below shows the parabolic path of a particle that is projected from the origin O with velocity V at an angle α to the horizontal. It lands at the point P, which lies on a plane inclined at an angle of β to the horizontal. When the particle strikes the plane, it is travelling at $90°$ to the plane. Let $OP = d$, and assume that the horizontal and vertical components of the displacement of the particle from O while it is moving on its parabolic path are given by
$x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$ where t is the time elapsed and g is the acceleration due to gravity.



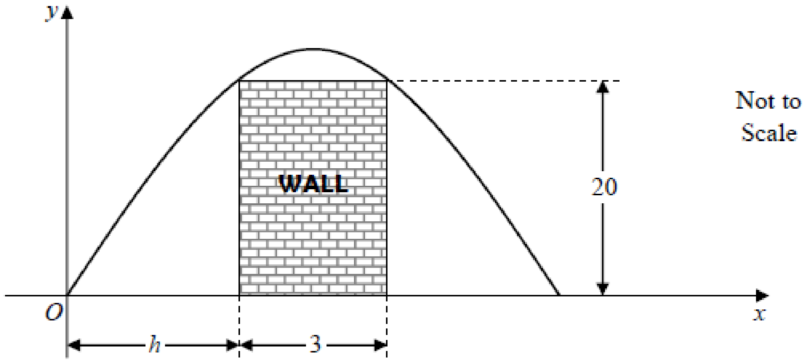
(i) Find the co-ordinates of P in terms of d and β. (1 mark)

(ii) By substituting the co-ordinates of P found in part (i) into the displacement equations, show that $d = \frac{2V^2\cos^2\alpha}{g\cos^2\beta}(\tan\alpha\cos\beta - \sin\beta)$. (2 marks)

(iii) By resolving the horizontal and vertical components of the velocity at P, show that $\cot\beta = \frac{gd\cos\beta}{V^2\cos^2\alpha} - \tan\alpha$. (3 marks)

(iv) Hence show that $\tan\alpha = \cot\beta + 2\tan\beta$. (2 marks)

9. The wall of a fort on level ground is 3 metres thick and 20 metres high. A projectile is fired from a point O outside the fort, h metres from the base of the wall of the fort, towards the fort as shown in the diagram below.



It is assumed that the path of the projectile traces out a parabola of the form $y = bx - ax^2$ where a and b are constants.

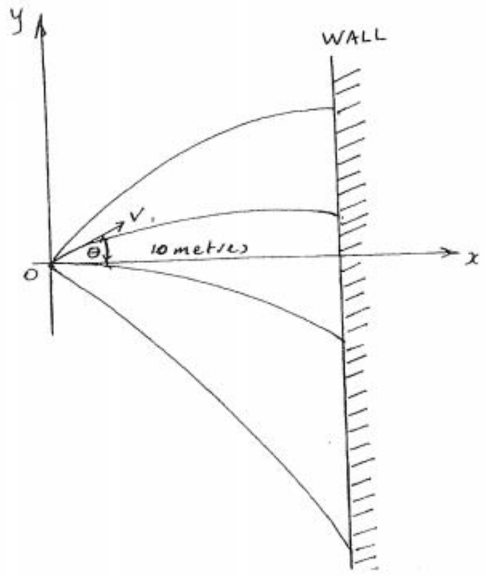
(i) Show that $b = \frac{20(2h+3)}{h(h+3)}$ and $a = \frac{20}{h(h+3)}$. (3 marks)

(ii) Let the angle of projection of the projectile be θ degrees and initial velocity be Vm/s and let $g = 10$m/s^2.
Hence, the equations of motion are $x = Vt\cos\theta$ and $= Vt\sin\theta - 5t^2$. Show that the equation of the path of flight of the projectile is given by: $y = x\tan\theta - \frac{5x^2}{V^2\cos^2\theta}$. (1 mark)

(iii) Hence show that $V^2\cos^2\theta = \frac{h(h+3)}{4}$. (1 mark)

(iv) If the projectile is fired at an angle of $45°$, find the values of h and V correct to two decimal places. (4 marks)

10. In the diagram below, a large number of projectiles are fired simultaneously from O, each with the same velocity V but various angles of elevation θ, at a wall distant 10 metres from O. The projectiles are fired so that their trajectories all lie in the same vertical plane perpendicular to the wall. (9 marks)



You may assume that the equations for the coordinates of a projectile at time t are $x = Vt\cos\theta$ and $y = -\frac{1}{2}gt^2 + Vt\sin\theta$.

(i) Use the identity $\sin^2\theta + \cos^2\theta = 1$ to eliminate θ from the two equations, and hence prove the relationship between height y and time t is:
$4y^2 + 4gt^2y + k = 0$ where $k = g^2t^4 + 4x^2 - 4V^2t^2$.

(ii) Show that the first impact on the wall occurs at time $t = \frac{10}{V}$ and that this projectile was fired horizontally. Also find where this projectile hits the wall.

(iii) Show that for $t > \frac{10}{V}$ there are two impacts at time t and that the distance between these impacts is $2\sqrt{V^2t^2 - 100}$.

(iv) Given that $V = 10$m/s, what are the initial angles of elevation of the two projectiles that strike the wall simultaneously $20\sqrt{3}$ metres apart.

End of homework