DUX College

Phone: (02) 8007 6824
Email: info@dc.edu.au
Web: dc.edu.au

2018 HIGHER SCHOOL CERTIFICATE COURSE MATERIALS

HSC Mathematics

Sequences and Series

Term 1 – Week 4

Name ……………………………………………………….

Class day and time ……………………………………………

Teacher name ………………………………………………...

Term 1 – Week 4 – Theory

SERIES APPLICATIONS:

EXAMPLE: (HOME LOAN REPAYMENT)

A person borrows $50,000 to purchase a home at 2% per month reducible interest and repays the loan in equal monthly installments over 5 years. What is the amount of each installment?

SOLUTION:

Let the amount owing after n months be A_n and the monthly installments be M

$A_1 = 50000 \times 1.02 - M$

$A_2 = A_1 \times 1.02 - M$

$= (50000 \times 1.02 - M) \times 1.02 - M$

$= 50000 \times 1.02^2 - M \times 1.02 - M$

$= 50000 \times 1.02^2 - M(1.02 + 1)$

$A_3 = A_2 \times 1.02 - M$

$= \left(50000 \times 1.02^2 - M(1.02 + 1)\right) \times 1.02 - M$

$= 50{,}000 \times 1.02^3 - M \times 1.02^2 - M \times 1.02 - M$

$= 50{,}000 \times 1.02^3 - M(1.02^2 + 1.02 + 1)$

Following this pattern, after 60 months

$A_{60} = 50{,}000 \times 1.02^{60} - M(1 + 1.02 + 1.02^2 + \cdots + 1.02^{59})$

$= 50000 \times 1.02^{60} - \frac{M(1.02^{60} - 1)}{1.02 - 1}$

But after 60 months all money are repaid. Thus $A_{60} = 0$,

$$0 = 50000 \times 1.02^{60} - \frac{M(1.02^{60} - 1)}{1.02 - 1}$$

$$\frac{M(1.02^{60} - 1)}{1.02 - 1} = 50000 \times 1.02^{60}$$

$$M = 50000 \times 1.02^{60} \times \frac{1.02 - 1}{1.02^{60} - 1}$$

$$\therefore M = 1438.40 \quad \text{(nearest cent)}$$

$\therefore$ The monthly installments are $1438.40 (to the nearest cent).

EXAMPLE: (HOME LOAN)

A person borrows $20,000 to purchase a home at 12% per annum reducible interest. He makes payments of $500 per month. Interest is calculated just before each payment.

(i) Find the amount owing after the first year.
(ii) Find the number of months required to repay the loan.

SOLUTION:

(i) Let A_n denote the amount owing after n months.

$$A_1 = 20000 \times 1.01 - 500$$

$$\begin{aligned} A_2 &= A_1 \times 1.01 - 500 \\ &= (20000 \times 1.01 - 500) \times 1.01 - 500 \\ &= 20000 \times 1.01^2 - 500 \times 1.01 - 500 \\ &= 20000 \times 1.01^2 - 500(1.01 + 1) \end{aligned}$$

$$\begin{aligned} A_3 &= A_2 \times 1.01 - 500 \\ &= \left(20000 \times 1.01^2 - 500(1.01 + 1)\right) \times 1.01 - 500 \\ &= 20000 \times 1.01^3 - 500(1.01^2 + 1.01 + 1) \end{aligned}$$

Following this pattern,

$$\begin{aligned} A_n &= 20000 \times 1.01^n - 500(1 + 1.01 + 1.01^2 + 1.01^3 + \cdots + 1.01^{n-1}) \\ &= 20000 \times 1.01^n - \frac{500(1.01^n - 1)}{1.01 - 1} \end{aligned}$$

Hence, after 1 year (i.e. $n = 12$)

$$A_{12} = 20000 \times 1.01^{12} - \frac{500(1.01^{12} - 1)}{1.01 - 1}$$

$\therefore A_{12} = 16195.25 \quad \text{(to th nearest cent)}$

$\therefore$ $16195.25 is still owing after 1 year.

(ii) $A_n = 0$

$$\begin{aligned} 20000 \times 1.01^n - \frac{500(1.01^n - 1)}{1.01 - 1} &= 0 \\ 20000 \times 1.01^n - 50000(1.01^n - 1) &= 0 \\ 20000(1.01)^n - 50000(1.01)^n + 50000 &= 0 \\ -30000(1.01)^n &= -50000 \\ (1.01)^n &= \frac{5}{3} \\ n &= \log_{1.01}\left(\frac{5}{3}\right) \\ n &= \frac{\ln\left(\frac{5}{3}\right)}{\ln 1.01} \\ n &= 51.34 \quad \text{(nearest 2 decimal places)} \\ \therefore n &= 52 \ \text{ months} \end{aligned}$$

$\therefore$ 52 months are required to repay the loan.

Term 1 – Week 4 – Homework

SERIES APPLICATIONS:

1. An investment fund intends to pay interest at the rate of 6% p.a. compounded every six months.
   (i) If an investment of \$250 is made today, what amount (i.e. principal and interest) will be available for withdrawal in ten years time?
   (ii) If nineteen further investments of \$250 are made every 6 months, show that the amount available for withdrawal in ten years time will be \$6919 to the nearest dollar. (Assume no withdrawals are made during this time)

2. Jasper borrowed $\$20\ 000$ from a finance company to purchase a car. Interest on the loan is calculated quarterly at a rate of 10% p.a. and is charged immediately prior to Jasper making his quarterly repayment of $\$M$.
   (i) Write an expression for A_1, the amount owing after 1 payment has been made.
   (ii) Show that $A_n = 20000 \text{ x } 1.025^n - 40M(1.025^n - 1)$.
   (iii) If the loan were to be paid out over 7 years, what would the value of M be?
   (iv) If Jasper were to pay $\$1282.94$ per quarter in repayments, how long would it take to pay out his loans?

3. When Jack left school he borrowed $\$15\ 000$ to buy his first car. The interest rate on the loan was 18% p.a. and Jack planned to pay back the loan in equal monthly instalments of $\$M$.
   (i) Show that immediately after making his first instalment, Jack owed $\$[15000 \text{ x } 1.015 - M]$.
   (ii) Show that immediately after making his third instalment, Jack owed $\$[15000 \text{ x } 1.015^3 - M(1 + 1.015 + 1.015^2)]$.
   (iii) Calculate the value of M.

4. An amount of $\$10\,000$ is borrowed and an interest rate of 1% per month is charged monthly. An amount M is repaid every month.
   (i) If $\$A_n$ is the amount owing after n months, show that
   $A_n = 100000(1.01)^n - M\left(\frac{1.01^n - 1}{0.01}\right)$.
   (ii) Find the value of M to the nearest cent, if the loan is to be repaid at the end of five years.
   (iii) How much extra in total will be repaid if the loan is repaid over a period of seven years?

5. Kelvin borrows $\$200\,000$ from his bank. Interest is compounded monthly at 0.425% per month. $\$A_n$ is the amount owed after n payments, $\$M$ is the amount of the monthly instalments and the loan is repaid after n months.
   (i) Show that $A_2 = 200000(1.00425)^2 - M(1.00425) - M$
   (ii) Show that $M = \frac{200000(1.00425)^n(0.00425)}{1.00425^n - 1}$
   (iii) Find the amount of the monthly instalments if Kelvin agrees to pay the loan over 30 years.
   (iv) If Kelvin instead decided to pay monthly instalments of $1331 from the beginning of the loan, how long will he take to repay the loan?

6. Simone borrows \$20 000 over 4 years at a rate of 1% per month, compounded monthly. If she pays of the loan in four equal yearly instalments, find:
   (i) The amount she will owe after 1 month.
   (ii) The amount she will owe after the first year, just before she pays the instalment.
   (iii) The amount of each instalment.
   (iv) The total amount of interest she will pay.

7. Karen borrows $\$15000$ from the bank. The loan plus interest and charges are to be repaid at the end of each month in equal monthly instalments of $\$M$ over five years. Interest is charged at 6% p.a. and is calculated on the balance owing at the beginning of each month. Furthermore, at the end of each month a bank charge of $\$15$ is added to the account. Let A_n be the amount owing after n months.
   (i) Write down expressions for A_1 and A_2 and show that the amount owing after three months is given by

$$A_3 = 15\,000 \text{ x } 1.005^3 - (M - 15)(1 + 1.005 + 1.005^2)$$

   (ii) Hence write an expression for A_n.
   (iii) Find the value of M correct to the nearest cent.

End of homework